Elastic cells in uniform flow

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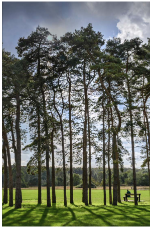


Applications

Unmanned aerial vehicles (UAVs)

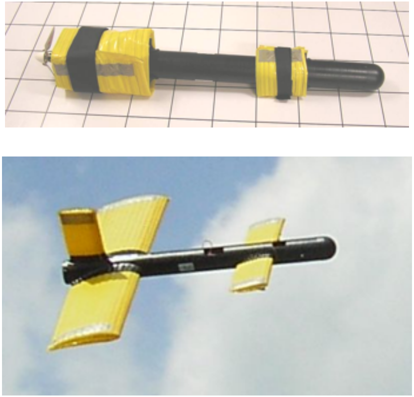
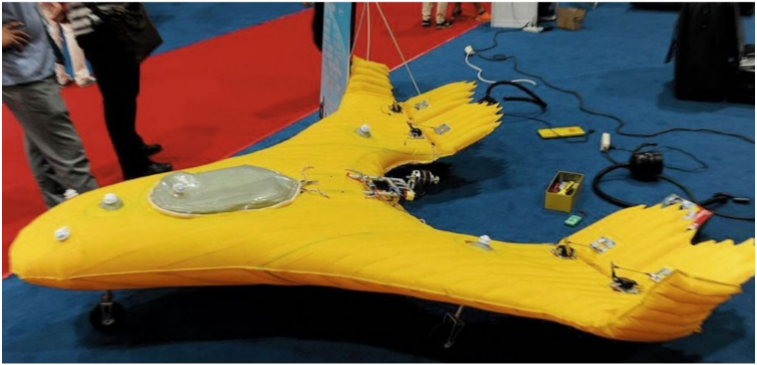


Figure: A UAV with inflatable elastic wings

- Advantages of UAV with inflatable wings: stowability, robustness,

...and inflatable drones



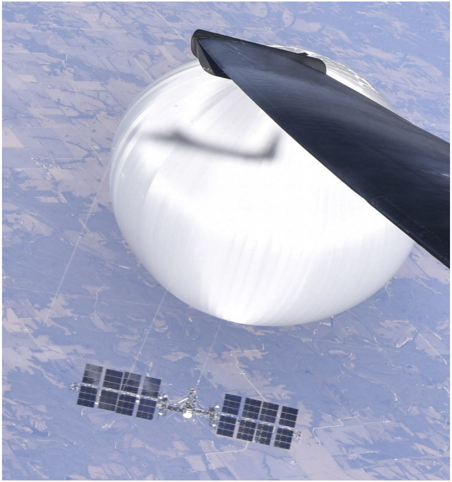
Advantages

Light, durable (likely to survive ground impact). Waterproof. Portability.

Review article

Li *et al.* (2018) A review of modelling and analysis of morphing wings

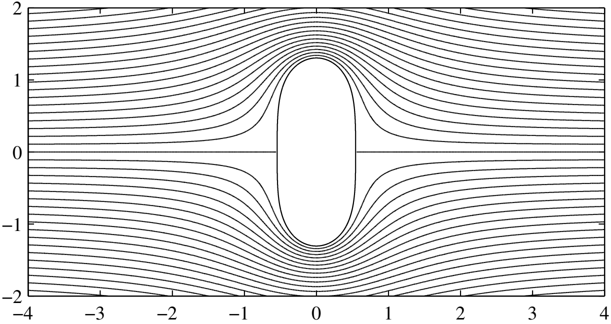
Chinese spy baloon



Wikipedia: 2023 Chinese balloon incident

Idealised Model

An elastic cell in a uniform stream



Inviscid, incompressible, irrotational flow

Related work

A two-dimensional bubble in a uniform stream

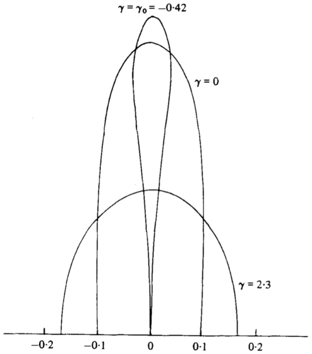


FIGURE 3. Same as figure 2 with $\beta = 90°$ and $\gamma_0 = -0{\cdot}42$. Each curve also represents half of a bubble in the absence of the wall.

[From: Vanden-Broeck & Keller (1980)]

Equilibrium states

– Vanden-Broeck & Keller (1980)

– Shankar (1992)

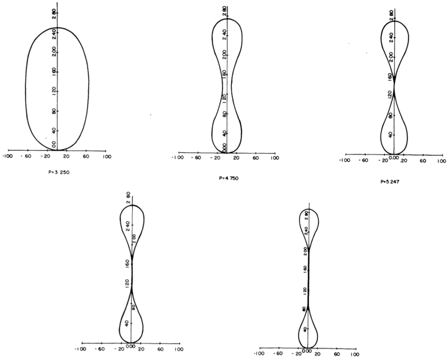
– Tanveer (1996)

Stability

– Nie & Tanveer (1995)

Related work

Static, pressurised elastic cell



[From: Flaherty *et al.* (1972)]

Buckled states

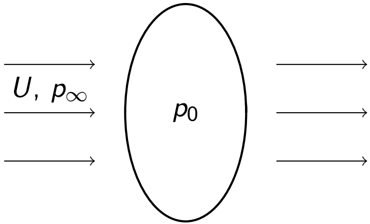
– Lévy (1884)

– Carrier (1947)

– Tadjbakhsh & Odeh (1967)

– Flaherty *et al.* (1972)

Cell in a uniform stream



Complex potential:

$$\frac{dw}{dz} = u - \mathrm{i}v, \qquad z = x + iy \qquad q = |w_z|$$

Far-field condition:

$$w \to Uz \quad \text{as} \quad z \to \infty$$

Bernoulli's equation:

$$\rho\,\mathrm{Re}\left(\frac{\partial w}{\partial t}\right) + \tfrac{1}{2}\rho q^2 + p = \tfrac{1}{2}\rho U^2 + p_\infty$$

Nondimensional governing equations

- Bernoulli's equation:

$$\mathrm{Re}\left(\frac{\partial w}{\partial t}\right) + \tfrac{1}{2}(q^2 - \alpha^2) - (\kappa_{ss} + \tfrac{1}{2}\kappa^3 - \sigma\kappa) - P = 0,$$

where $\alpha = \sqrt{\frac{L^3 \rho U^2}{8\pi^3 E_B}}$, $P = \frac{8\pi^3 L^3 (p_\infty - p_0)}{E_B}$.

- Inextensibility

$$\eta(s + 2\pi) = \eta(s),$$

- Kinematic condition

$$\boldsymbol{\eta}_t \cdot \hat{\mathbf{n}} = \mathbf{u} \cdot \hat{\mathbf{n}}.$$

- Far-field condition:

$$w \to \alpha z \quad \text{as} \quad |z| \to \infty$$

Steady state problem (no flow)

Flaherty *et al.* (1972)

($\alpha = 0$)

$$\kappa_{ss} + \frac{1}{2}\kappa^3 - \sigma\kappa + P = 0$$

Buckling pressures

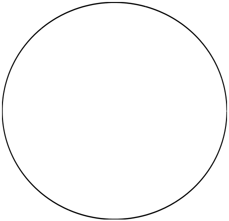
$$P = n^2 - 1 \qquad n = 2, 3, 4, \dots$$

Contact pressures

$$P = 5.247, \qquad P = 21.65, \qquad P = 51.84$$

Buckling pressures

$P = n^2 - 1$ for $n = 2, 3, 4, \cdots$

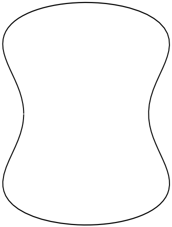


Pre-buckling pressure

$P < 3$

Buckling pressures

$P = n^2 - 1$ for $n = 2, 3, 4, \cdots$

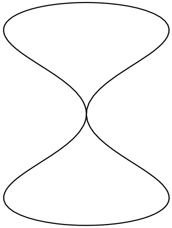


Mode 2 buckling

$P > 3$

Buckling pressures

$P = n^2 - 1$ for $n = 2, 3, 4, \cdots$

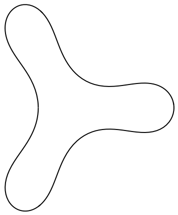


Mode 2 self-contact

$P = 5.247$

Buckling pressures

$P = n^2 - 1$ for $n = 2, 3, 4, \cdots$

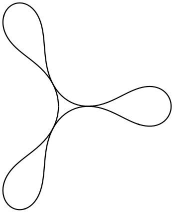


Mode 3 buckling

$P > 8$

Buckling pressures

$P = n^2 - 1$ for $n = 2, 3, 4, \cdots$

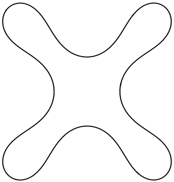


Mode 3 self-contact

$P = 21.65$

Buckling pressures

$P = n^2 - 1$ for $n = 2, 3, 4, \cdots$

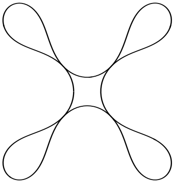


Mode 4 buckling

$P > 15$

Buckling pressures

$P = n^2 - 1$ for $n = 2, 3, 4, \cdots$

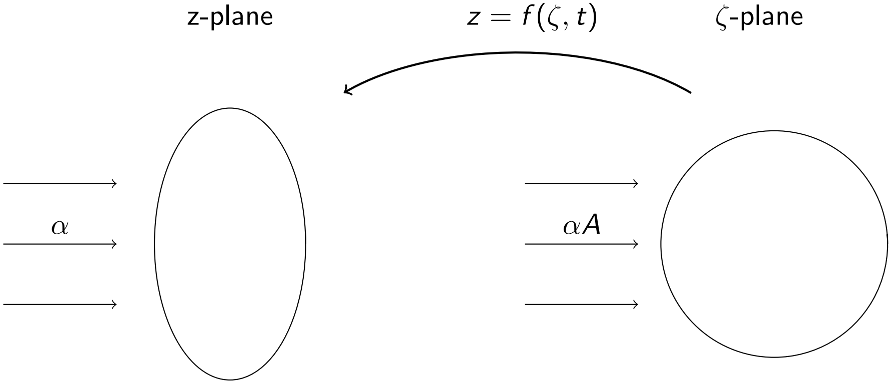


Mode 4 self-contact

$P = 51.84$

Conformal mapping

We seek a mapping from the flow around a circle (Shankar 1992)



Conformal mapping

We take the mapping functions to be

$$z = A(t)\zeta + a_0(t) + \sum_{n=1}^{\infty} a_n(t)\zeta^{-n},$$

$$w = \alpha A(t)\zeta + b_0(t) + \sum_{n=1}^{\infty} b_n(t)\zeta^{-n},$$

with the values on the cell wall given by the parametrisation $\zeta = e^{i\phi}$

$$\eta(\phi, t) = A(t)e^{i\phi} + a_0(t) + \sum_{n=1}^{\infty} a_n(t)e^{-in\phi},$$

$$\Omega(\phi, t) = \alpha A(t)e^{i\phi} + b_0(t) + \sum_{n=1}^{\infty} b_n(t)e^{-in\phi},$$

Governing equations

- Bernoulli's equation

$$\mathrm{Re}\left(\frac{\partial\Omega}{\partial t} - \frac{\Omega_\phi}{\eta_\phi}\frac{\partial\eta}{\partial t}\right) = -\tfrac{1}{2}(q^2 - \alpha^2) + \kappa_{ss} + \tfrac{1}{2}\kappa^3 - \sigma\kappa + P,$$

- Kinematic condition

$$\mathrm{Im}(\overline{\eta}_\phi \frac{\partial\eta}{\partial t}) = -\,\mathrm{Im}(\Omega_\phi)$$

- Inextensibility:

$$\int_0^{2\pi} |\eta_\phi| d\phi = 2\pi.$$

Steady solutions

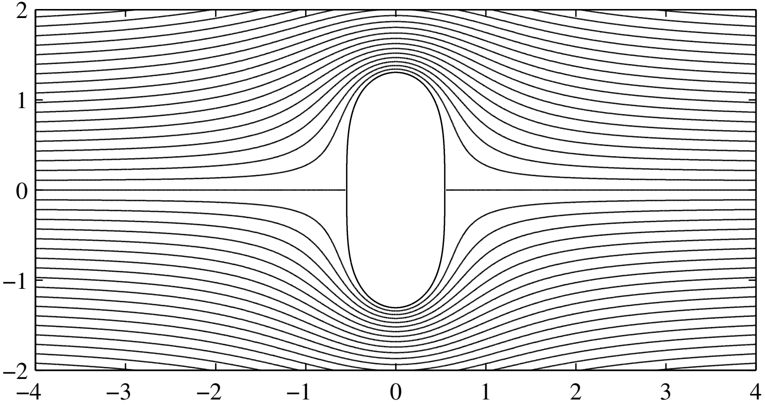


Figure: The vertically oriented steady solution at $\alpha = 1$, $P = 2$

Steady solutions

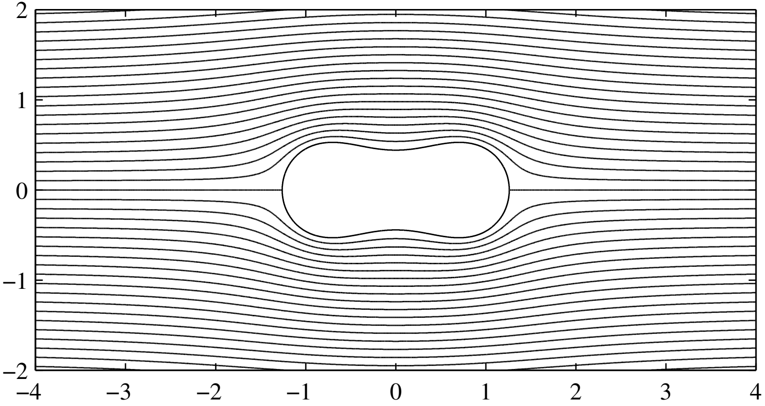


Figure: The horizontally oriented steady solution at $\alpha = 1$, $P = 4$

Steady solutions

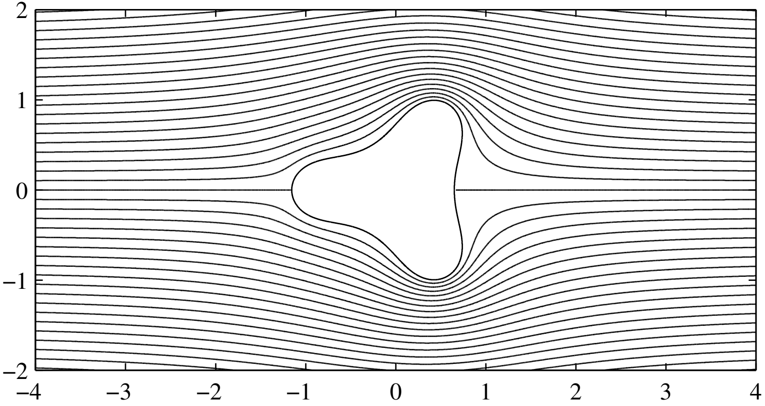


Figure: An asymmetric solution at $\alpha = 1$, $P = 9$

Steady solutions

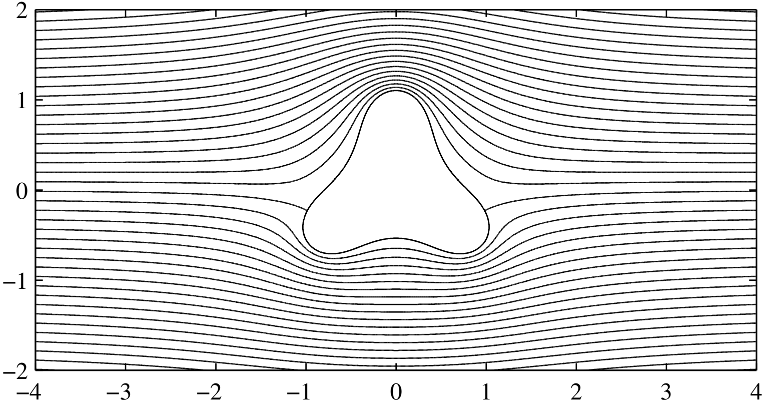


Figure: An asymmetric solution at $\alpha = 1$, $P = 9$

Steady solutions

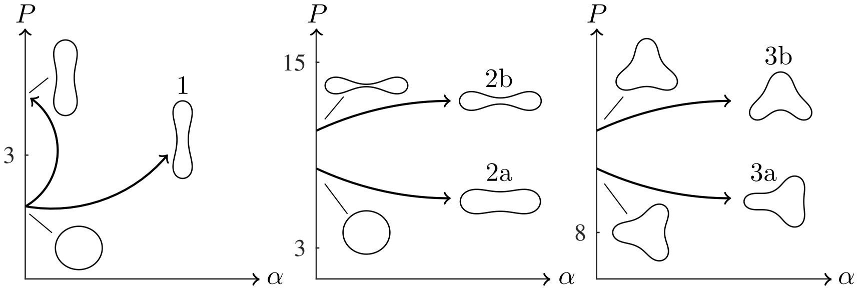


Figure: Sketch showing the classification of the steady solutions. At $\alpha = 0$, Flaherty *et al.* (1972). The arrows illustrate a continuous path taken to reach a particular cell type.

Linear stability os the steady solutions

To study the linear stability of the steady solutions, we introduce a small perturbation to both the cell boundary and the flow itself

$$\eta(\phi, t) = \eta^s(\phi) + \hat{\eta}(\phi, t), \quad \Omega(\phi, t) = \Omega^s(\phi) + \hat{\Omega}(\phi, t)$$

Taking a small perturbation $\hat{\mathbf{x}}(t) = \{\hat{a}_{1\ldots n}, \hat{b}_{1\ldots n}\}$, we obtain

$$A\hat{\mathbf{x}}_t = B\hat{\mathbf{x}},$$

where A and B depend on the steady terms a_n^s, b_n^s calculate above.
Taking $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0 e^{\lambda t}$, we have

$$(A\lambda - B)\,\hat{\mathbf{x}}_0 = 0.$$

A steady solution is said to be spectrally stable if $Re(\lambda) \leq 0$ for all eigenvalues λ , and unstable otherwise.

Linear stability

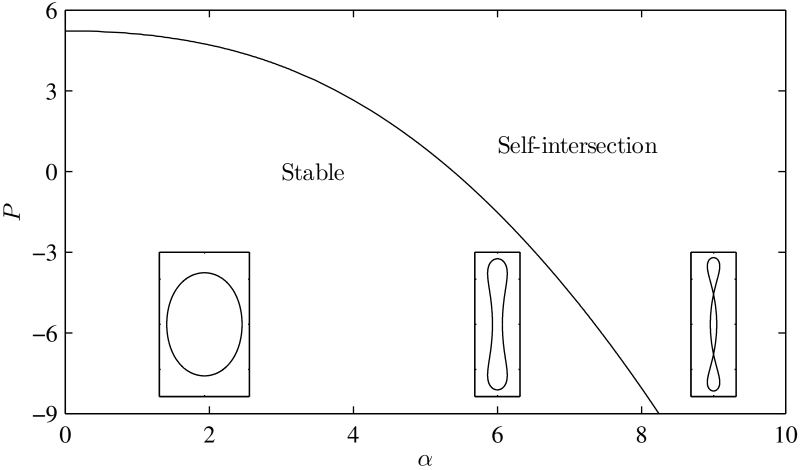


Figure: Properties of the vertically oriented cells.

Linear stability

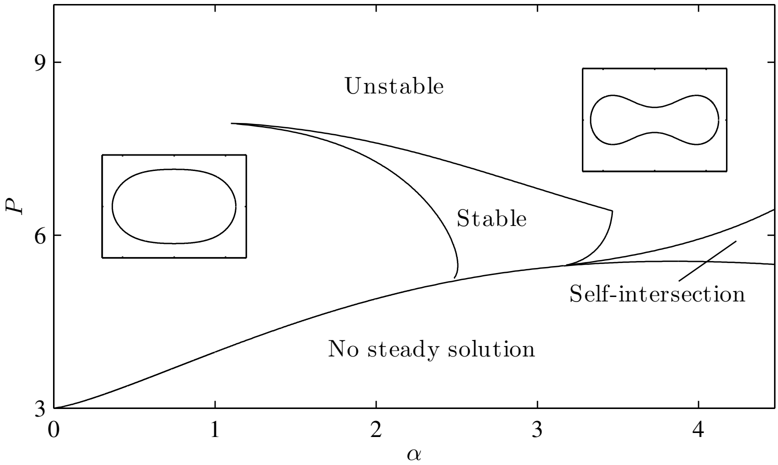


Figure: Properties of the horizontally oriented cells

Time-evolution of the fully nonlinear unsteady system

The governing equations are

$$\mathrm{Im}(\overline{\eta}_\phi \frac{\partial \eta}{\partial t}) = -\,\mathrm{Im}(\Omega_\phi)$$

$$\mathrm{Re}\left(\frac{\partial \Omega}{\partial t}\right) = \mathrm{Re}\left(\frac{\Omega_\phi}{\eta_\phi}\frac{\partial \eta}{\partial t}\right) - \tfrac{1}{2}(q^2 - \alpha^2) + \kappa_{ss} + \tfrac{1}{2}\kappa^3 - \sigma(t)\kappa + P,$$

$$\int_0^{2\pi} \kappa\,\mathrm{Im}(\Omega_\phi) d\phi = 0.$$

where

$$\eta = A(t)e^{in\phi} + a_0(t) + \sum_{n=1}^{\infty} a_n(t)e^{-in\phi}$$

$$\Omega = \alpha A(t)e^{in\phi} + b_0(t) + \sum_{n=1}^{\infty} b_n(t)e^{-in\phi},$$

Numerical method

Using a Hilbert transform, we obtain an explicit system.
Kinematic condition:

$$\eta_t = \mathrm{i}\eta_\phi \,\mathrm{Re}\left(\frac{\mathrm{i}\Omega_\phi}{|\eta_\phi|^2}\right) + \eta_\phi \mathcal{H}\left(\mathrm{Re}\left(\frac{\mathrm{i}\Omega_\phi}{|\eta_\phi|^2}\right)\right),$$

Bernoulli's equation:

$$\begin{aligned}\Omega_t =& \frac{\Omega_\phi}{\eta_\phi}\eta_t - \tfrac{1}{2}(q^2 - \alpha^2) + (\kappa_{ss} + \tfrac{1}{2}\kappa^3) + P \\ &+ \mathrm{i}\mathcal{H}\left(\tfrac{1}{2}u^2 - \kappa_{ss} - \tfrac{1}{2}\kappa^3\right) + \sigma\left(\mathrm{i}\mathcal{H}\left(\kappa\right) - \kappa\right),\end{aligned}$$

Inextensibility condition:

$$\int_0^{2\pi} \kappa_t \,\mathrm{Im}(\Omega_\phi) + \kappa \,\mathrm{Im}(\Omega_{\phi,t}) d\phi = 0.$$

Unstable eigenvalues for horizontally oriented cells

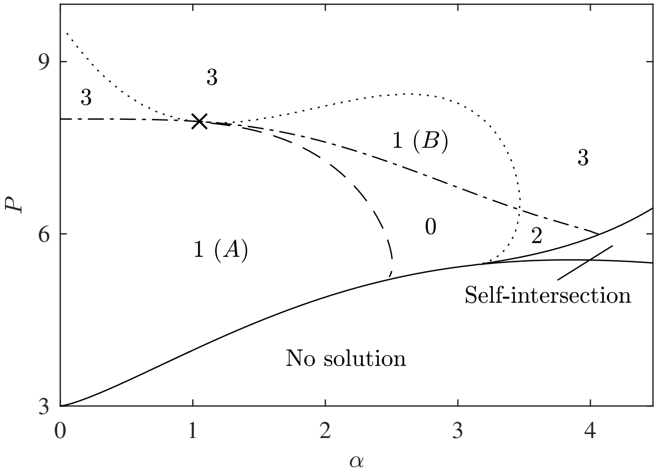
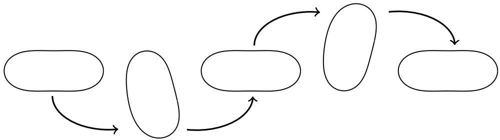


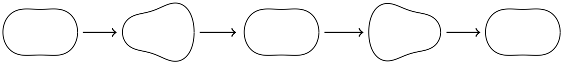
Figure: Properties of the horizontally oriented cells

Unsteady results: horizontally oriented cells

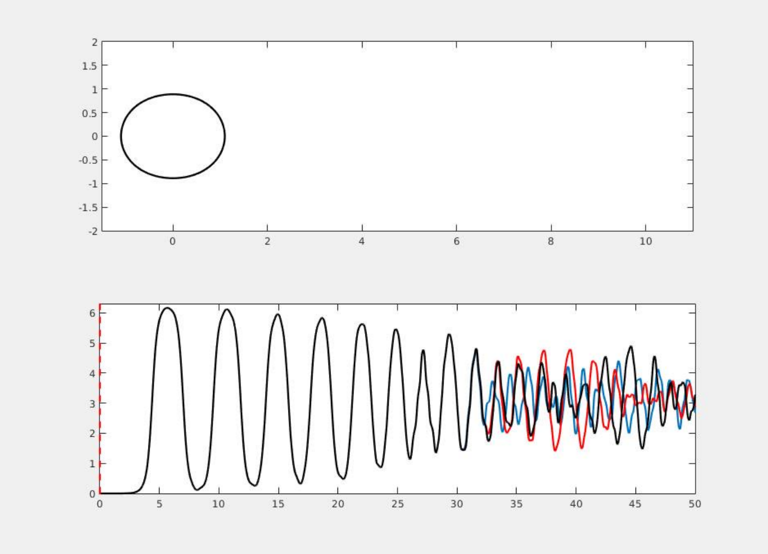
- (A) $\alpha = 2$, $P = 6$



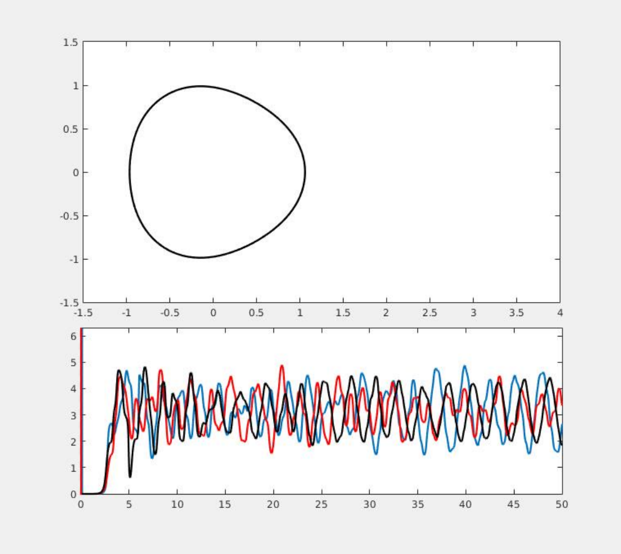
- (B) $\alpha = 2.5$, $P = 8$



Unsteady results: $\alpha = 1$, $P = 6$



Unsteady results: $\alpha = 0.5$, $P = 8.05$



Cells with corners

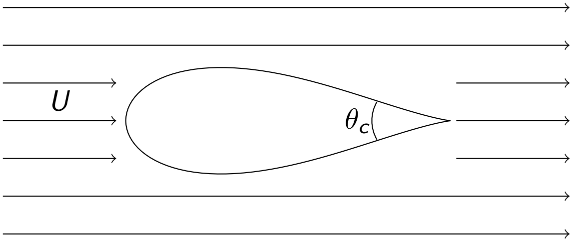


Figure: A sketch of the flow

$$q = 0 \quad \text{at} \quad s = 0, \qquad \text{Kutta condition}$$

$$w - \alpha z - \frac{\beta}{2\pi i} \log(z) \to 0 \quad \text{as} \quad z \to \infty, \qquad \text{far-field condition}$$

$$\tfrac{1}{2}(q^2 - \alpha^2) - (\kappa_{ss} + \tfrac{1}{2}\kappa^3 - \sigma\kappa) - P - \frac{\alpha\beta}{2\pi}(x_s + \kappa y) = 0.$$

with dimensionless parameters $\alpha = \sqrt{\frac{\ell^3 \rho U^2}{E_B}}, \beta = \Gamma\sqrt{\frac{\ell\rho}{E_B}}$.

Equilibria (no flow)

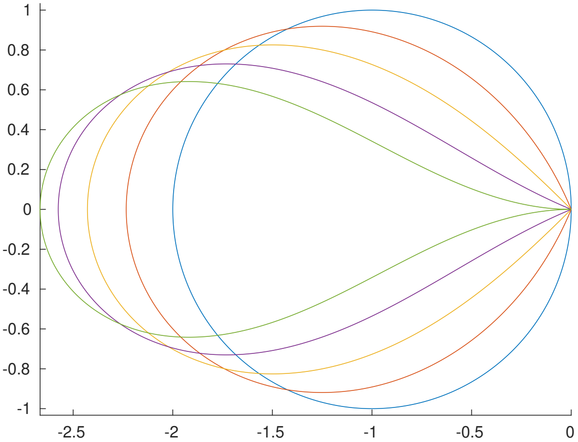
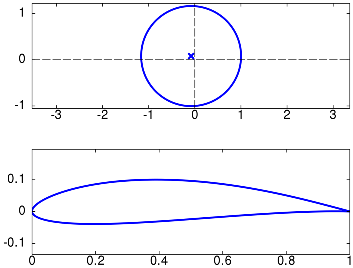


Figure: Cell subject to a uniform external pressure for various corner angles

Karman-Trefftz mapping for elastic aerofoil in uniform flow

ζ-plane (circle) $\rightarrow$ z-plane (aerofoil)

$$z = p\frac{(\zeta+1)^p + (\zeta-1)^p}{(\zeta+1)^p - (\zeta-1)^p}, \qquad p = 2 - \frac{\theta_c}{\pi}$$



Cell with corner ($\beta \neq 0$) in the uniform flow

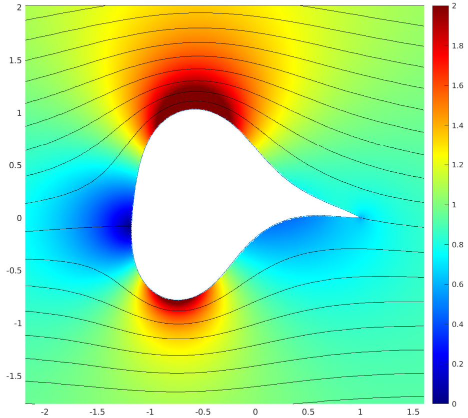
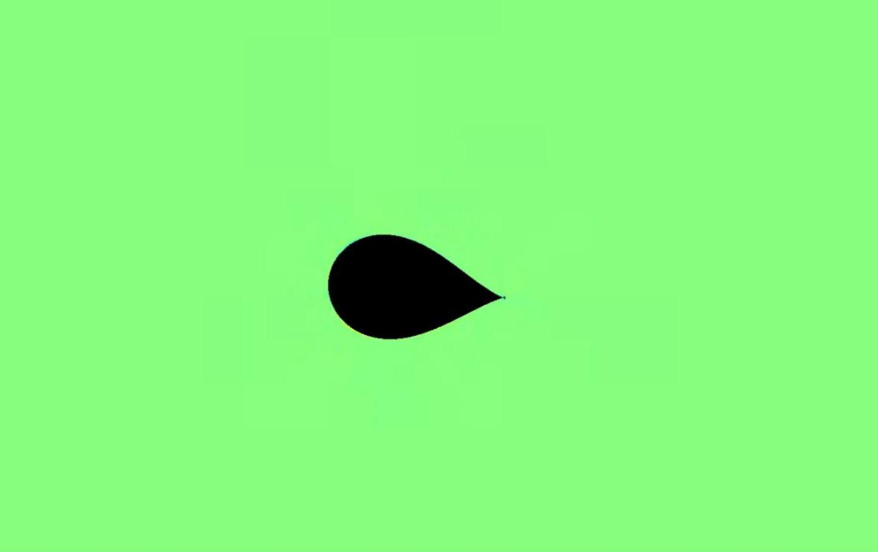


Figure: Potential flow for $\alpha = 2$, $P = 5$, $\beta = 3$, $\theta_c = \pi/6$

Viscous simulations



Gerris simulation for the viscous flow, with $Re = 10^4$, $\beta = \alpha = 1$, $P = 0$.

Other work

- Consider elastic cells with internal support

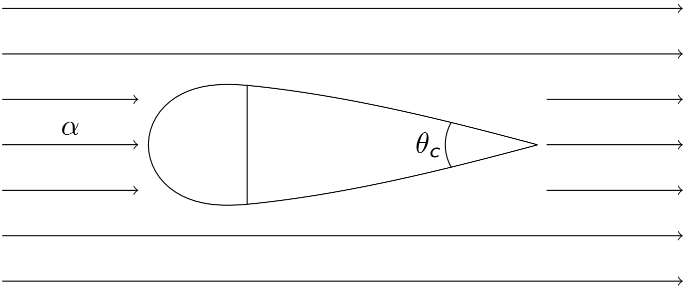


Figure: Sketch of the flow

References

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2. A. Yorkston, M. G. Blyth, E. I. Părău, The deformation and stability of an elastic cell in a uniform flow, *SIAM J. Appl. Math.* **80**(1), (2020), 71–94.

3. A. Yorkston, M. G. Blyth, E. I. Părău, A model of an inflatable elastic aerofoil, *J. Engng. Maths.* **131**, 11 (2021).

4. A. A. Yorkston, M. G. Blyth, E. I. Părău, The deformation of an elastic cell in a circulatory fluid motion, *Wave Motion* **13** (2022), 102995