Extensions for operators on Hilbert spaces which satisfy polynomial growth conditions

LAURIAN SUCIU

∼ Results obtained in collaboration with ∼

CĂTĂLIN BADEA
Lille

VLADIMIR MÜLLER
Prague

AURELIAN CRĂCIUNESCU
Timișoara

GENERAL CONSIDERATIONS

Denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. Let $\mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{K}$. We denote by $P_{\mathcal{H}} \in \mathcal{B}(\mathcal{K})$ the orthogonal projection onto $\mathcal{H}$.

Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$.

We say that S is an **extension** of T if $S\mathcal{H} \subset \mathcal{H}$ and $S|_{\mathcal{H}} = T$, i.e. on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, the operator S has the form

$$S = \begin{pmatrix} T & \star \\ 0 & \star \end{pmatrix}.$$

We say that S is a (power) **dilation** of T if

$$T^n = P_{\mathcal{H}} S^n|_{\mathcal{H}}, \quad \forall\ n \geq 0.$$

This is equivalent with one of the following matrix representations

$$S = \begin{pmatrix} T & \star \\ 0 & \star \end{pmatrix}, \quad \text{or} \quad S = \begin{pmatrix} \star & \star & \star \\ 0 & T & \star \\ 0 & 0 & \star \end{pmatrix}, \quad \text{or} \quad S = \begin{pmatrix} \star & \star \\ 0 & T \end{pmatrix}.$$

The existence of unitary dilations for Hilbert space contractions are basic results in dilation theory (see the monographs of **Sz.-Nagy-Foias-Bercovici-Kerchy, Foias-Frazho, N. K. Nikolski** [23, 10, 18]).

Recall that $T \in \mathcal{B}(\mathcal{H})$ is called ***m*-isometric** for some $m \geq 1$ if it satisfies the relation

$$\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*j}T^j = 0.$$

1-isometries are just isometries, (see the trilogy of **Agler-Stankus** [1, 2, 3] for more about m-isometries).

The powers of an m-isometry S can grow only polynomially: $\exists$ K such that

$$\|S^n\|^2 \leq Kn^{m-1}, \forall\ n \in \mathbb{N}.$$

Therefore any T which has an m-isometric dilation must satisfy the same estimate.

In particular T is a **2-isometry** if $T^{*2}T^2 - 2T^*T + I = 0$. Also, T is called:

- **concave** if $T^{*2}T^2 - 2T^*T + I \leq 0$ (i.e. $(\|T^n x\|^2)_{n\geq 0}$ is a concave sequence for any $x \in \mathcal{H}$);
- **convex** if $T^{*2}T^2 - 2T^*T + I \geq 0$ (i.e. $(\|T^n x\|^2)_{n\geq 0}$ is a convex sequence for any $x \in \mathcal{H}$);
- **expansive** if $T^*T - I \geq 0$. A concave operator is expansive.

For a given bounded sequence $(\lambda_n)_{n=0}^{\infty} \subseteq \mathbb{C}$ there exists a unique operator $W \in \mathcal{B}(\ell^2(\mathcal{H}))$, called a **unilateral weighted shift** with weights $(\lambda_n)_{n=0}^{\infty}$, such that

$$W(h_0, h_1, h_2, \cdots) = (0, \lambda_0 h_0, \lambda_1 h_1, \cdots), \quad n \in \mathbb{N}.$$

Theorema 1.1

If $m \in \mathbb{N}^*$ and $T \in \mathcal{B}(\mathcal{H})$, then the following conditions are equivalent :

(i) *T is an m-isometry,*

(ii) *$T^{*n}T^n$ is a polynomial in n of degree at most $m-1$,*

(iii) *for each $h \in \mathcal{H}$, $\|T^n h\|^2$ is a polynomial in n of degree at most $m-1$,*

(iv) *T is injective and for each nonzero $h \in \mathcal{H}$, the unilateral weighted shift $W_{T,h} \in \mathcal{B}(\ell^2(\mathbb{C}))$ with weights $\left(\frac{\|T^{n+1}h\|}{\|T^n h\|}\right)_{n=0}^{\infty}$ is an m-isometry.*

Agler-Stankus, 1995

$B \in \mathcal{B}(\mathcal{H})$ is called a **Brownian unitary operator** if

$$B = \begin{pmatrix} V & \sigma E \\ 0 & U \end{pmatrix},$$

where
$\rightarrow$ V, E are isometries with $V^*E = 0$ and $\mathrm{Ran}(E) = \mathrm{Ker}(V^*)$;
$\rightarrow$ U is unitary;
$\rightarrow$ $\sigma^2 = \|B^*B - I\|$, where $\Delta_B = B^*B - I$ is the covariance operator for B.

If T is a 2-isometry on $\mathcal{H}$ then there exist $\mathcal{K} \supset \mathcal{H}$ and B on $\mathcal{K}$ a Brownian unitary with the same covariance as T such that $B|_{\mathcal{H}} = T$. Hence an operator which has a 2-isometric dilation has also a Brownian unitary dilation.

Let T be a left invertible operator on $\mathcal{H}$, $T' = T(T^*T)^{-1}$ its Cauchy dual. T' is also left invertible.

We define

$$\mathcal{H}_\infty(T) = \bigcap_{n\geq 0} T^n\mathcal{H}.$$

We say that T is

$\rightarrow$ **analytic** if $\mathcal{H}_\infty(T) = \{0\}$;

$\rightarrow$ has **the wandering subspace property (WSP)** if $\bigvee_{n\geq 0} T^n \mathrm{Ker}(T^*) = \mathcal{H}$;

$\rightarrow$ admits a **Wold type decomposition (WTD)** if

$$\mathcal{H} = \mathcal{H}_\infty(T) \oplus \bigvee_{n\geq 0} T^n \mathrm{Ker}(T^*),$$

where the subspaces are reducing for T and $T|_{\mathcal{H}_\infty(T)}$ is unitary.

S. Shimorin, 2001

T is **analytic** iff T' has **WSP**;
T admits a **WTD** iff T' admits a **WTD**. In this case $\mathcal{H}_\infty(T) = \mathcal{H}_\infty(T')$.
If T is concave then T is **analytic** iff T' is **analytic**. Also, if T is concave then it admits a **WTD**.
If T is **analytic** then

$$\mathcal{H} \longleftrightarrow \mathcal{D} = \{\Theta_h : h \in \mathcal{H}\}$$

$$h \leftrightarrow \Theta_h, \quad \Theta_h : D(0, r(T')^{-1}) \to \mathrm{Ker}(T^*)$$

$$(\Theta_h)(z) = \sum_{n\geq 0} (P_{\mathrm{Ker}(T^*)} T'^{*n} h) z^n$$

$$T \longleftrightarrow M_z \quad \text{on} \quad \mathcal{D}, \quad (M_z f)(z) = zf(z)$$

$$T'^* \longleftrightarrow B_z \quad \text{on} \quad \mathcal{D}, \quad (B_z f)(z) = \frac{f(z) - f(0)}{z}.$$

A. Olofsson, 2004

If T is an analytic 2-isometry then $\mathcal{D} = \mathcal{D}_\mu$ where

$$\mu : \mathrm{Bor}(\mathbb{T}) \to \mathcal{B}(\mathrm{Ker}(T^*));$$

$$\widehat{\mu}(n) = \widehat{\mu}(-n)^* = P_{\mathrm{Ker}(T^*)} T^{*n}(T^*T - I)|_{\mathrm{Ker}(T^*)}; \quad n \geq 0;$$

$$\|f\|_\mu^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} \langle P(\mu)(z) f'(z), f'(z) \rangle \mathrm{d}A(z);$$

$$P(\mu)(z) = \int_{\mathbb{T}} P(z, e^{i\theta}) \mathrm{d}\mu(e^{i\theta}), \quad z \in \mathbb{D}$$

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}, \quad z \in \mathbb{D}.$$

m-ISOMETRIC DILATIONS

Theorema 2.1

Let $m \geq 0$ be an integer and let $T \in \mathcal{B}(\mathcal{H})$ be an operator satisfying the condition

$$\sup_{n\geq 1} n^{-m/2}\|T^n\| < \infty. \tag{2.1}$$

Then T has an expansive and analytic $(m+3)$-isometric dilation.

Proof.

Suppose first that the Hilbert space $\mathcal{H}$ is separable.
Let $K \geq \max\{1, n^{-m/2}\|T^n\| \ : n \geq 1\}$. Then

$$\|T^n\|^2 \leq K^2 n^m, \quad n \geq 1.$$

For every integer $s \geq 1$ we set

$$\alpha_s = \left(\frac{2Ks+1}{2K(s-1)+1} \right)^{(m+2)/2}.$$

Clearly $\alpha_1 \geq \alpha_2 \geq ... \geq 1$.
Let $\ell^2_+(\mathcal{H}) = \bigoplus_{j=0}^{\infty} \mathcal{H}_j$, where $\mathcal{H}_j = \mathcal{H}$ for $j \geq 0$, and let S be the weighted forward shift of multiplicity $\dim \mathcal{H}$ with the weights α_s, i.e., S is defined by

$$S(h_0, h_1, ...) = (0, \alpha_1 h_0, \alpha_2 h_1, ...)$$

for all sequences $(h_0, h_1, ...) \in \ell^2_+(\mathcal{H})$. Then

$$\|S^n(h_0, 0, ...)\|^2 = \|(0, 0, ..., (2Kn+1)^{(m+2)/2} h_0, 0, ...)\|^2 = (2Kn+1)^{m+2}.$$

Moreover, it is easy to see that **S is an $(m+3)$-isometry.**

Proof.

Let S^* be the adjoint of S, i.e., S^* is the weighted backward shift defined by

$$S^*(h_0, h_1, h_2, \ldots) = (\alpha_1 h_1, \alpha_2 h_2, \ldots).$$

We prove now that **S^* is (unitarily equivalent to) an extension of T^***. Indeed, for $s \geq 1$, let

$$b_s = (\alpha_1 \cdots \alpha_s)^{-2} = (2Ks+1)^{-m-2}.$$

Using (2.1), we get

$$\begin{aligned}\sum_{s=1}^{\infty} b_s \|T^{*s}\|^2 &= \sum_{s=1}^{\infty} b_s \|T^s\|^2 \leq K^2 \sum_{s=1}^{\infty} s^m (2Ks+1)^{-m-2} \\ &\leq K^{-m} 2^{-m-2} \sum_{s=1}^{\infty} s^{-2} \leq \frac{\pi^2}{24} < 1.\end{aligned}$$

Thus, by **V. Müller** [17, Theorem 2.2], T^* **is unitarily equivalent to a restriction of** S^* **to an invariant subspace** ($\mathcal{H}$ being separable).

In conclusion S is an $(m+3)$-isometric dilation of T and it is clear that S is analytic and expansive (because $\alpha_s \geq 1$ for all $s \geq 1$). □

Remark 2.2

We have the following implications:

$$\textbf{T has m-isometric dilation} \quad \Longrightarrow \quad \sup_n \frac{\|T^n\|^2}{n^{m-1}} < \infty$$

$$\Longrightarrow \quad \textbf{T has an expansive, analytic and minimal (m+2)-isometric dilation}.$$

Invertible m-isometric extensions. If T is an invertible m-isometry and m is even, then T is an $(m-1)$-isometry. Suppose that $m+3$ is odd.

The $(m+3)$-isometric operator S in Theorem 2.1 has an invertible $(m+3)$-isometric extension $\widehat{S}$.

Indeed, assuming that

$$\|T^n\|^2 \leq K^2 n^m, \quad n \geq 1,$$

for fixed m and K, set $w_n = (2Kn+1)^{m+2}$ for $n \in \mathbb{Z}$.

Let $\hat{S}$ be the weighted bilateral shift of multiplicity $\dim \mathcal{H}$ defined by

$$\widehat{S}(\dots, h_{-1}, h_0, h_1, \dots) = \left(\dots, \sqrt{\frac{w_{-1}}{w_{-2}}}h_{-2}, \sqrt{\frac{w_0}{w_{-1}}}h_{-1}, \sqrt{\frac{w_1}{w_0}}h_0, \dots\right).$$

Clearly $\widehat{S}$ is invertible and $(m+3)$-isometric. Moreover, $\widehat{S}$ is a dilation of T.

Corollary 2.3

Every power bounded operator has an invertible 3-isometric dilation.

Since every invertible 2-isometry is a unitary operator Corollary 2.3 is optimal.

In the case of Foguel-Hankel type operators, using a result of **Bermudez-Martinon-Müller-Noda** [6] we can give the following

Theorema 2.4

Let $T \in \mathcal{B}(\mathcal{H})$ be an operator such that, with respect to an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, has the block matrix form

$$T = \begin{pmatrix} C_0 & E \\ 0 & C_1 \end{pmatrix},$$

where C_i are contractions on $\mathcal{H}_i$ ($i = 0, 1$) and $E \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ is such that $EC_1 = C_0 E$.

Then T has a 3-isometric dilation on $\mathcal{K} \supset \mathcal{H}$

$$S = \begin{pmatrix} V_0 & L \\ 0 & V_1 \end{pmatrix} = \begin{pmatrix} V_0 & 0 \\ 0 & V_1 \end{pmatrix} + \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}.$$

where V_i are the minimal isometric dilations of C_i, $i = 0, 1$ and L is a dilation for E such that $LV_1 = V_0 L$.

Furthermore, S can be extended to a **Jordan operator** *J i.e $J = U + N$, U unitary, $N^2 = 0$ and $UN = NU$ (see [16]).*

Corollary 2.5

Every Foguel-Hankel operator, i.e. $T = \begin{pmatrix} S_+^* & X \\ 0 & S_+ \end{pmatrix}$ where $XS_+ = S_+^* X$, S_+ being the unilateral shift can be dilated to a Jordan operator.

SUB-BROWNIAN m-ISOMETRIES AND THEIR EXTENSIONS

In what follows we investigate a class of m-isometries which have Brownian type extensions in the sense of the definition below. We refer here to m-isometries $T \in \mathcal{B}(\mathcal{H})$ for an integer $m \geq 3$ that is with $\Delta_T^{(m)} = 0$, which are $\Delta_T^{(j)}$**-bounded** for $j = 1, 2, ..., m-2$, where

$$\Delta_T^{(1)} = \Delta_T = T^*T - I \quad \text{and} \quad \Delta_T^{(j+1)} = T^*\Delta_T^{(j)}T - \Delta_T^{(j)}.$$

This means that $\Delta_T^{(j)} \geq 0$ and there exist constants $\alpha_j > 0$ such that

$$T^*\Delta_T^{(j)}T \leq \alpha_j^2 \Delta_T^{(j)}, \quad j = 1, 2, ..., m-2. \tag{3.1}$$

In this case the conditions (3.1) are equivalent to

$$0 \leq \Delta_T^{(j+1)} \leq (\alpha_j^2 - 1)\Delta_T^{(j)}, \quad j = 1, 2, ..., m-2. \tag{3.2}$$

For T, j satisfying (3.1) let $\sigma_j \geq 1$ be the scalar given by

$$\sigma_j := \inf\{\alpha > 1 : T^*\Delta_T^{(j)}T \leq \alpha^2 \Delta_T^{(j)}\}. \tag{3.3}$$

Then the scalar

$$\sigma := \max\{\|\Delta_T^{1/2}\|, \quad (\sigma_j^2 - 1)^{1/2}; \quad j = 1, 2, ..., m-2\} \tag{3.4}$$

is called the **covariance** of T, and it is denoted as $\sigma = \mathrm{cov}(T)$.

We illustrate now examples of operators satisfying the conditions of the form (3.1).

An operator $B \in \mathcal{B}(\mathcal{H})$ is called an ***m*-Brownian unitary** for an integer $m \geq 2$, if under a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_m$, B has a matrix representation of the form

$$B = \begin{pmatrix} V_1 & \delta E_1 & 0 & ... & 0 & 0 \\ 0 & V_2 & \delta E_2 & ... & 0 & 0 \\ .. & .. & .. & ... & .. & .. \\ 0 & 0 & 0 & ... & V_{m-1} & \delta E_{m-1} \\ 0 & 0 & 0 & ... & 0 & U \end{pmatrix}, \tag{3.5}$$

where V_j, E_j are isometries with $\mathcal{N}(V_j^*) = \mathcal{R}(E_j)$ for $j = 1, 2, ..., m-1$, U is unitary and $\delta > 0$ is a scalar.

The following main result shows that the m-Brownian unitaries play the same role in the theory of m-isometries as $(2-)$ Brownian unitaries in the context of 2-isometries (see [2, Theorem 5.80]).

Theorema 3.1

For an operator $T \in \mathcal{B}(\mathcal{H})$ a scalar $\sigma > 0$ and an integer $m \geq 3$, the following statements are equivalent:

(i) *T is m-isometric and $\Delta_T^{(j)}$-bounded for $j = 1, 2, ..., m-2$ with $\mathrm{cov}(T) \leq \sigma$;*

(ii) *T has an extension to an m-Brownian unitary B with $\mathrm{cov}(B) = \sigma$.*

An *m*-isometry satisfying (3.1) is called a **sub-Brownian *m*-isometry.**

Proof.

Let T be as in (i), that is satisfying the conditions:

$$\Delta_T^{(m)} = 0, \quad \|\Delta_T\| \leq \sigma^2, \quad \Delta_T^{(j)} \geq 0, \quad T^*\Delta_T^{(j)}T \leq (\sigma^2+1)\Delta_T^{(j)}$$

for $j = 1, 2, ..., m-2$. Denote shortly $\Delta_1 = \Delta_T$ and $\Delta_j = \Delta_T^{(j)}$ for $j = 2, ..., m$. So we have

$$I - \sigma^{-2}\Delta_1 \geq 0, \quad \Delta_{j-1} - \sigma^{-2}\Delta_j \geq 0 \quad (j = 2, ..., m).$$

Now from the last condition we obtain for $j \in \{2, ..., m-1\}$,

$$T^*(\Delta_{j-1} - \sigma^{-2}\Delta_j)T - \Delta_{j-1} + \sigma^{-2}\Delta_j = \Delta_j - \sigma^{-2}\Delta_{j+1} \geq 0$$

therefore $T^*(\Delta_{j-1} - \sigma^{-2}\Delta_j)T \geq \Delta_{j-1} - \sigma^{-2}\Delta_j$. On the other hand, using the fact that $T^*\Delta_1 T \leq (\sigma^2+1)\Delta_1$ we get the relation

$$T^*(I - \sigma^{-2}\Delta_1)T = T^*T - \sigma^{-2}T^*\Delta_1 T \geq \Delta_1 + I - (1+\sigma^{-2})\Delta_1 = I - \sigma^{-2}\Delta_1.$$

This together with the previous inequalities provide that there exist the contractions C_1' from $\mathcal{R}[(I - \sigma^{-2}\Delta_1)^{1/2}T]$ into $\mathcal{R}[(I - \sigma^{-2}\Delta_1)^{1/2}]$ and C_j' from $\mathcal{R}[(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}T]$ into $\mathcal{R}[(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}]$ for $j \in \{2, ..., m-1\}$, such that

$$C_1'(I - \sigma^{-2}\Delta_1)^{1/2}T = (I - \sigma^{-2}\Delta_1)^{1/2}, \quad C_j'(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}T = (\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}.$$

Proof.

Next, these contractions C_j' for $j \in \{1, 2, ..., m-1\}$ can be extended (by continuity and orthogonality) to some contractions $C_j^* \in \mathcal{B}(\mathcal{H}_j)$, where

$$\mathcal{H}_1 = \overline{\mathcal{R}(I - \sigma^{-2}\Delta_1)}, \quad \mathcal{H}_j = \overline{\mathcal{R}(\Delta_{j-1} - \sigma^{-2}\Delta_j)},$$

such that $C_1^* = 0$ on $\mathcal{H}_1 \ominus \overline{\mathcal{R}[(I - \sigma^{-2}\Delta_1)^{1/2}T]}$ and $C_j^* = 0$ on $\mathcal{H}_j \ominus \overline{\mathcal{R}[(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}T]}$ for $j \in \{2, ..., m-1\}$. So we have the relations

$$(I - \sigma^{-2}\Delta_1)^{1/2} = T^*(I - \sigma^{-2}\Delta_1)^{1/2}C_1, \quad (\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2} = T^*(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}C_j,$$

which lead to the identities

$$T^*(I-\sigma^{-2}\Delta_1)^{1/2}(I-C_1C_1^*)(I-\sigma^{-2}\Delta_1)^{1/2}T = T^*(I-\sigma^{-2}\Delta_1)T-(I-\sigma^{-2}\Delta_1) = \Delta_1-\sigma^{-2}\Delta_2$$

and respectively

$$T^*(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}(I - C_jC_j^*)(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}T =$$

$$T^*(\Delta_{j-1} - \sigma^{-2}\Delta_j)T - (\Delta_{j-1} - \sigma^{-2}\Delta_j) = \Delta_j - \sigma^{-2}\Delta_{j+1}.$$

Proof.

Now for $j \in \{1, 2, ..., m-1\}$ let V_j' on $\mathcal{K}_j' \supset \mathcal{H}_j$ be an isometric dilation for C_j. So $V_j'^*|_{\mathcal{H}_j} = C_j^*$ and denoting $\mathcal{N}_j = \mathcal{N}(V_j'^*)$ we have that

$$I - C_j C_j^* = P_{\mathcal{H}_j}(I - V_j' V_j'^*)|_{\mathcal{H}_j} = P_{\mathcal{H}_j} P_{\mathcal{N}_j}|_{\mathcal{H}_j},$$

where $P_{\mathcal{H}_j}, P_{\mathcal{N}_j} \in \mathcal{B}(\mathcal{K}_j')$ are the corresponding orthogonal projections. Now the previous identities for C_j permit to define the isometries E_j' from $\mathcal{H}_{j+1}$ into $\mathcal{N}_j$ with $\mathcal{R}(E_j') \subset \mathcal{N}_j$, such that

$$E_1'(\Delta_1 - \sigma^{-2}\Delta_2)^{1/2} h = P_{\mathcal{N}_1}(I - \sigma^{-2}\Delta_1)^{1/2} Th,$$

and respectively

$$E_j'(\Delta_j - \sigma^{-2}\Delta_{j+1})^{1/2} h = P_{\mathcal{N}_j}(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2} Th,$$

for $h \in \mathcal{H}$ and $j = 2, ..., m-1$. Clearly, the isometry E_{m-1}' from $\mathcal{H}_m = \overline{\mathcal{R}(\Delta_{m-1})}$ into $\mathcal{N}_{m-1}$ satisfies the relation

$$E_{m-1}'(\Delta_{m-1}^{1/2} h) = P_{\mathcal{N}_{m-1}}(\Delta_{m-2} - \sigma^{-2}\Delta_{m-1})^{1/2} Th.$$

Notice that if for an index j one has $\mathcal{R}(E_j') \neq \mathcal{N}_j$ then $\mathcal{E}_j = \mathcal{N}_j \ominus \mathcal{R}(E_j')$ is a wandering subspace for V_j' i.e. $V_j'^n \mathcal{E}_j \perp V_j'^q \mathcal{E}_j$ for $n, q \geq 0$, $n \neq q$, while the subspace $\ell^2_+(\mathcal{E}_j) = \bigoplus_{n=0}^{\infty} V_j'^n \mathcal{E}_j$ of $\mathcal{K}_j'$ is reducing for V_j'. In this case $\widetilde{V}_j = V_j'|_{\mathcal{K}_j' \ominus \ell^2_+(\mathcal{E}_j)}$ is an isometric dilation for C_j with $\mathcal{N}(\widetilde{V}_j^*) = \mathcal{R}(E_j')$. Thus to simplify the notation we can assume that $V_j' = \widetilde{V}_j$, so that $\mathcal{N}_j = \mathcal{R}(E_j')$, for $j \in \{1, 2, ..., m-1\}$.

Proof.

Next we have in view that $T^*\Delta_{m-1}T = \Delta_{m-1}$ (T being an m-isometry), hence there exists an isometry V on $\mathcal{H}_m = \overline{\mathcal{R}(\Delta_{m-1})}$ such that

$$V\Delta_{m-1}^{1/2} = \Delta_{m-1}^{1/2}T.$$

Let U on $\mathcal{K}_m \supset \mathcal{H}_m$ be a unitary extension for V. Consider the spaces

$$\mathcal{K}_{m-1} = \mathcal{L}_{m-1} \oplus \mathcal{K}'_{m-1}, \quad \text{where} \quad \mathcal{L}_{m-1} = \ell^2_+(\mathcal{K}_m \ominus \mathcal{H}_m),$$

and successively for $j = m-2, ..., 2, 1$, the spaces

$$\mathcal{K}_j = \mathcal{L}_j \oplus \mathcal{K}'_j, \quad \text{where} \quad \mathcal{L}_j = \ell^2_+(\mathcal{K}_{j+1} \ominus \mathcal{H}_{j+1}).$$

Let S_j be the forward shift on $\mathcal{L}_j$, so $\mathcal{N}(S_j^*) = \mathcal{K}_{j+1} \ominus \mathcal{H}_{j+1}$. Define the mappings $V_j = S_j \oplus V'_j$ on $\mathcal{K}_j = \mathcal{L}_j \oplus \mathcal{K}'_j$ and $E_j : \mathcal{K}_{j+1} \to \mathcal{K}_j$, this later having the block matrix

$$E_j = \begin{pmatrix} 0 & L_j \\ E'_j & 0 \end{pmatrix} : \begin{bmatrix} \mathcal{H}_{j+1} \\ \mathcal{K}_{j+1} \ominus \mathcal{H}_{j+1} \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_j \\ \mathcal{K}'_j \end{bmatrix},$$

where $L_j : \mathcal{K}_{j+1} \ominus \mathcal{H}_{j+1} \to \mathcal{L}_j$ is the embedding mapping. Then V_j is an isometric dilation for C_j, while E_j is an isometry from $\mathcal{K}_{j+1}$ into $\mathcal{K}_j$ with $\mathcal{N}(V_j^*) = \mathcal{R}(E_j)$, for $j = 1, 2, ..., m-1$.

Proof.

Now we are able to define the desired extension of T. This is the m-Brownian unitary on $\mathcal{K} = \bigoplus_{j=1}^{m} \mathcal{K}_j$ with the representation

$$B = \begin{pmatrix} V_1 & \delta E_1 & 0 & \dots & 0 & 0 \\ 0 & V_2 & \delta E_2 & \dots & 0 & 0 \\ .. & .. & .. & \dots & .. & .. \\ 0 & 0 & 0 & \dots & V_{m-1} & \delta E_{m-1} \\ 0 & 0 & 0 & \dots & 0 & U \end{pmatrix}. \tag{3.6}$$

To prove that B is an extension for T we find an isometry $Z : \mathcal{H} \to \mathcal{K}$ which satisfies the relation $ZTh = BZh$ for $h \in \mathcal{H}$. Thus we define Z by the relation

$$Zh = (I - \sigma^{-2}\Delta_1)^{1/2} h \oplus \left(\bigoplus_{j=2}^{m} \sigma^{-(j-1)} (\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2} h \right)$$

for $h \in \mathcal{H}$. It is easy to see for $j = 2, 3, ..., m-1$ that

$$\|\sigma^{-(j-1)}(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2} h\|^2 = \sigma^{-2(j-1)} \|\Delta_{j-1}^{1/2} h\|^2 - \sigma^{-2j} \|\Delta_j^{1/2} h\|^2$$

and $\|\sigma^{-(m-1)}\Delta_{m-1}^{1/2} h\|^2 = \sigma^{-2(m-1)}\|\Delta_{m-1}^{1/2} h\|^2$ (for $j = m$). So it follows that $\|Zh\|^2 = \|h\|^2$ for $h \in \mathcal{H}$, that is Z is an isometry.

Proof.

Also, we have the relations

$$ZTh = (I - \sigma^{-2}\Delta_1)^{1/2} Th \oplus \bigoplus_{j=2}^{m} \sigma^{-j+1}(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2} Th,$$

and by (3.6),

$$\begin{aligned} BZh &= [V_1(I - \sigma^{-2}\Delta_1)^{1/2}h + E_1(\Delta_1 - \sigma^{-2}\Delta_2)^{1/2}h] \\ &\oplus \bigoplus_{j=2}^{m-1}[V_j(\sigma^{-j+1}(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}h + \sigma E_j(\sigma^{-j}(\Delta_j - \sigma^{-2}\Delta_{j+1})^{1/2}h)] \\ &\oplus \sigma^{-m+1}U\Delta_{m-1}^{1/2}h = [V_1(I - \sigma^{-2}\Delta_1)^{1/2}h + E_1(\Delta_1 - \sigma^{-2}\Delta_2)^{1/2}h] \\ &\oplus \bigoplus_{j=2}^{m-1} \sigma^{-j+1}[V_j(\Delta_{j-1} - \sigma^{-2}\Delta_j)^{1/2}h + E_j(\Delta_j - \sigma^{-2}\Delta_{j+1})^{1/2}h] \oplus \sigma^{-m+1}U\Delta_{m-1}^{1/2}h. \end{aligned}$$

The last terms of ZT and BZ (for $j = m$) coincide, having in view that $\Delta_{m-1}^{1/2}Th = V\Delta_{m-1}^{1/2}h = U\Delta_{m-1}^{1/2}h$. For the other terms of ZT and BZ we use that $V_j^*|_{\mathcal{H}_j} = C_j^*$, as well as the definitions of C_j (resp. C_j') and E_j, for $j = 1, 2, ..., m-1$.

Proof.

Thus we have the relations

$$\begin{aligned}(I-\sigma^{-2}\Delta_1)^{1/2}Th &= V_1C_1^*(I-\sigma^{-2}\Delta_1)^{1/2}Th+(I-V_1V_1^*)(I-\sigma^{-2}\Delta_1)^{1/2}Th\\ &= V_1(I-\sigma^{-2}\Delta_1)^{1/2}h+E_1(\Delta_1-\sigma^{-2}\Delta_2)^{1/2}h,\end{aligned}$$

and respectively for $j=2,3,...,m-1$,

$$\begin{aligned}(\Delta_{j-1}-\sigma^{-2}\Delta_j)^{1/2}Th &= V_jC_j^*(\Delta_{j-1}-\sigma^{-2}\Delta_j)^{1/2}Th+(I-V_jV_j^*)(\Delta_{j-1}-\sigma^{2}\Delta_j)^{1/2}Th\\ &= V_j(\Delta_{j-1}-\sigma^{-2}\Delta_j)^{1/2}h+E_j(\Delta_j-\sigma^{-2}\Delta_{j+1})^{1/2}h.\end{aligned}$$

These identities show that $ZT=BZ$, so the subspace $Z\mathcal{H}=\bigoplus_{j=1}^m\mathcal{H}_j\subset\mathcal{K}$ is invariant for B. Since Z is unitary from $\mathcal{H}$ onto $Z\mathcal{H}$ we conclude that T is unitarily equivalent to $B|_{Z\mathcal{H}}$. In other words, this means that B is an extension for T. Thus we proved that (i) implies (ii).

The converse implication is immediate. Indeed, if B is as m-Brownian unitary extension for T with $\mathrm{cov}(B)=\sigma$ then $\Delta_T^{(j)}=P_{\mathcal{H}}\Delta_B^{(j)}|_{\mathcal{H}}$ for $j=1,2,...,m$. So $\Delta_T^{(m)}=0$ i.e. T is an m-isometry and $\|\Delta_T\|\leq\|\Delta_B\|=\sigma^2$. Also, since

$$T^*\Delta_T^{(j)}T=P_{\mathcal{H}}B^*\Delta_B^{(j)}B|_{\mathcal{H}}\leq(\sigma^2+1)P_{\mathcal{H}}\Delta_B^{(j)}|_{\mathcal{H}}=(\sigma^2+1)\Delta_T^{(j)},$$

we infer that $\sigma\geq(\sigma_j^2-1)^{1/2}$ where σ_j is given by (3.6), for $j=1,2,...,m-2$. Hence $\mathrm{cov}(T)\leq\sigma$, which shows that (ii) implies (i). □

From this result and Theorem 2.1 we have the following

Corollary 3.2

If $T \in \mathcal{B}(\mathcal{H})$ is an operator which for an integer $m \geq 3$ satisfies the condition

$$\sup_{n \geq 1} n^{-\frac{m-3}{2}} \|T^n\| < \infty,$$

then T has an m-Brownian unitary dilation.

In particular, if T is power bounded then it has a 3-Brownian unitary dilation.

Theorema 3.3

For a non-isometric operator $T \in \mathcal{B}(\mathcal{H})$ and an integer $m \geq 3$ the following statements are equivalent:

(i) *T is a sub-Brownian m-isometry;*

(ii) *T is expansive and there exists a sub-Brownian $(m-1)$-isometry $W \in \mathcal{B}(\mathcal{H})$ such that*

$$\Delta_T^{1/2} T = W \Delta_T^{1/2}.$$

We characterize now the sub-Brownian m-isometric weighted shifts.

Theorema 3.4

Let p be a polynomial with complex coefficients of degree $m-1$ for an integer $m \geq 3$, such that $p(n) > 0$ for each integer $n \geq 0$. Let S_m on $\mathcal{K} = \ell^2_+(\mathcal{H})$ be the weighted shift with weights $(\lambda_n)_{n\geq 0}$, where $\lambda_n = \sqrt{\frac{p(n+1)}{p(n)}}$ for $n \geq 0$. Then S_m is a sub-Brownian m-isometry if and only if the polynomial p satisfies the conditions

$$p_q(n) := \sum_{j=0}^{q} (-1)^j \binom{q}{j} p(n+q-j) > 0 \tag{3.7}$$

for all integers $n \geq 0$ and $q = 1, 2, ..., m-2$, with $p_{m-2}(1) > p_{m-2}(0)$.
In particular this happens when all coefficients of p are positive.

Corollary 3.5

Let S on $\mathcal{K} = \ell^2_+(\mathcal{H})$ be the 3-isometric weighted shift with weights $(\lambda_n)_{n\geq 0}$, where $\lambda_n = \sqrt{\frac{p(n+1)}{p(n)}}$ and $p(n) = an^2 + bn + c > 0$, for $n \geq 0$ and some scalars $a \neq 0$, b and c. Then S is a sub-Brownian 3-isometry if and only if $a > 0$ and $a + b > 0$.

Theorema 3.6

Let $T \in \mathcal{B}(\mathcal{H})$.

(i) *If T is a* **convex operator** *such that the sequence $\left(\frac{T^n}{\sqrt{n}}\right)_n$ is bounded then it has an extension $\widetilde{T}$ on a Hilbert space $\mathcal{M} \supset \mathcal{H}$ with $\widetilde{T}$ of the form*

$$\widetilde{T} = \begin{pmatrix} C & E \\ 0 & U \end{pmatrix}$$

on a decomposition $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$, where:
$\rightarrow$ C is a contraction, U is unitary and there exists F on $\mathcal{M}' \supset \mathcal{D}_C = \overline{\mathrm{Ran}(I - C^*C)^{1/2}} = \overline{\mathrm{Ran}(D_C)}$ such that

$$\overline{\mathrm{Ran}} \begin{pmatrix} D_C \\ C \end{pmatrix} \perp \overline{\mathrm{Ran}} \begin{pmatrix} F \\ E \end{pmatrix}.$$

(ii) *If T is a* **concave operator** *then it has an extension $\widetilde{T}$ on a Hilbert space $\mathcal{M}$ which on $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ has the form*

$$\widetilde{T} = \begin{pmatrix} V & E \\ 0 & U \end{pmatrix},$$

$\rightarrow$ with V an isometry, U a unitary operator and $V^*E = 0$.

REFERENCES I

- Agler J., Stankus M. m-isometric transformations of Hilbert spaces. Integral Equations Operator Theory, 1995, **21** (4) 383–429.
- Agler J., Stankus M. m-isometric transformations of Hilbert spaces II. Integral Equations Operator Theory, 1995, **23** (10) 1–48.
- Agler J., Stankus M. m-isometric transformations of Hilbert spaces III. Integral Equations Operator Theory, 1996, **24** 379–421.
- C. Badea, V. Müller, L. Suciu, *High order isometric liftings and dilations*, Studia Math., **258** (1), (2021), 87–101.
- C. Badea, L. Suciu, *Hilbert space operators with two-isometric dilations*, J. Oper. Theory **86** (1), (2021), 93–123.
- T. Bermúdez, A. Martinón, V. Müller, J. A. Noda, *Perturbation of m-Isometries by Nilpotent Operators*, Abstract and Applied Analysis, **2014**, Article ID 745479, 6 pages, 2014.
- A. Biswas, A. E. Frazho, C. Foias, *Weighted commutant lifting*, Acta Sci. Math. (Szeged), **65**, 3–4 (1999), 657–686.
- G. Cassier, *Generalized Toeplitz operators, restrictions to invariant subspaces and similarity problems*, J. Operator Theory, **53:1** (2005), 49-89.
- A. Crăciunescu, L. Suciu, *Brownian extensions in the context of 3-isometries*, (2023), 1-22, submitted.

REFERENCES II

- Foias F., Frazho A. E. The Commutant Lifting Approach to Interpolation Problems. Birkhäuser Verlag, Basel-Boston-Berlin, 1990.
- L. Kérchy, *Generalized Toeplitz operators*, Acta Sci. Math. (Szeged) **68** (2002), 373-400.
- C.S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997.
  1067-1090.
- W. Majdak, L. Suciu, *Triangulations of operators with two-isometric liftings*, Integral Equations and Operator Theory, (2021) **93:10**, 1-24.
- W. Majdak, L. Suciu, *Convex and expansive liftings close to two-isometries and power bounded operators*, Linear Algebra and its Applications, **617** (2021), 1–26.
- McCullough S. SubBrownian operators. J Operator Theory, 1989, **22** 291–305.
- S. McCullough, B. Russo, *The 3-Isometric Lifting Theorem*, Integral Equations Operator Theory **84**, 1 (2016), 69–87.
- V. Müller, *Models for operators using weighted shifts*, J. Operator Th. **20**, 1 (1988), 3-20.
- N. K. Nikolski, *Operators, Functions and Systems: An Easy Reading*, vol I,II, AMS 2002.
- Olofsson A. A von Neumann-Wold decomposition of two-isometries. Acta Sci Math (Szeged), 2004, **70** 715–726.

REFERENCES III

Richter S. A representation theorem for cyclic analytic two-isometries. Trans Amer Math Soc, 1991, **328** 325–349.

Shimorin S. Wold-type decompositions and wandering subspaces for operators close to isometries. J Reine Angew Math, 2001, **531** 147–189.

Suciu L. On operators with two-isometric liftings. Complex Analysis and Operator Theory, 2020, **14:5**, 1-16.

Sz.-Nagy B., Foias C., Bercovici H., Kérchy L. Harmonic Analysis of Operators on Hilbert Space. Revised and enlarged edition. In: Universitext, Springer, New York, 2010.