On canonical parameterizations of curves

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This talk is about
INFINITE-DIMENSIONAL GEOMETRY

- ✗ Kähler and hyperkähler Banach manifolds
- ✗ Banach Poisson–Lie groups
- ✓ Shape spaces of curves

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The aim of the talk is to
DESCRIBE CANONICAL SECTIONS OF THE FIBER BUNDLE CONSISTING OF PARAMETERIZED CURVES

Why infinite-dimensional geometry?

- because most shape spaces are infinite-dimensional
- natural objects on a finite-dimensional manifold are elements of an infinite-dimensional space (vector fields, Riemannian metrics, mesures...)
- existence of geodesics on a finite-dimensional manifold is an infinite-dimensional phenomenon
  - initial value problem or shooting : geodesic is a solution of a Cauchy problem, i.e. a fixed point of a contraction in an appropriate infinite-dimensional space of curves
  - 2 boundary value problem: geodesic is a curve minimising an energy functional on a infinite-dimensional space of curves
- Each time one want to vary the geometry of a finite-dimensional manifold, one ends up with a infinite-dimensional manifold (of Riemannian metric, of connexions, of symplectic forms....)

Application: Research on Rheumatoid Arthritis

Rheumatoid Arthritis

Rheumatoid Arthritis is a long-term autoimmune disorder that primarily affects joints leading to pain and stiffness. The causes of this disease are not clear. The quantification of disease progression is crucial in order to adapt the treatment to the patient.

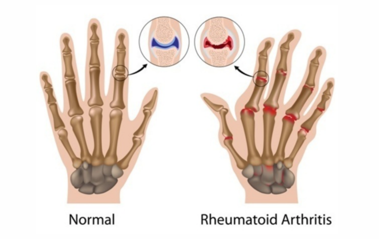
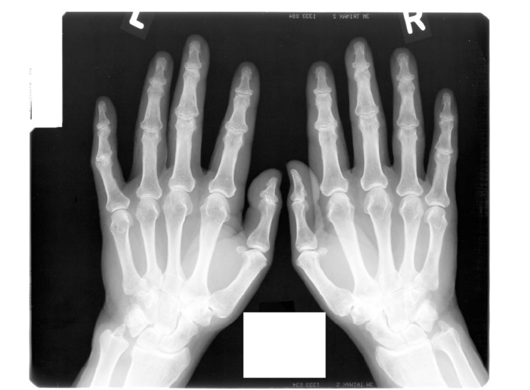


Figure: credit: [2]

Rheumatoid Arthritis

X-ray modality is used to detect and quantify Rheumatoid Arthritis.
X-Ray projective radiography creates black and white images of bones.



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Rheumatoid Arthritis

Key features to detect are Erosion and Joint Space shrinking.

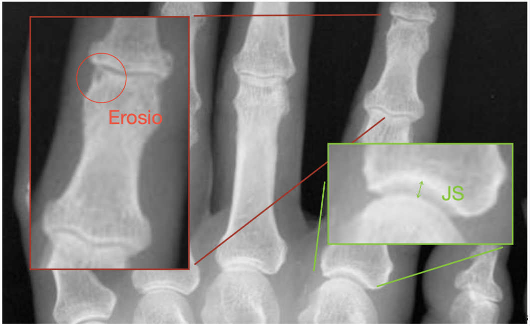
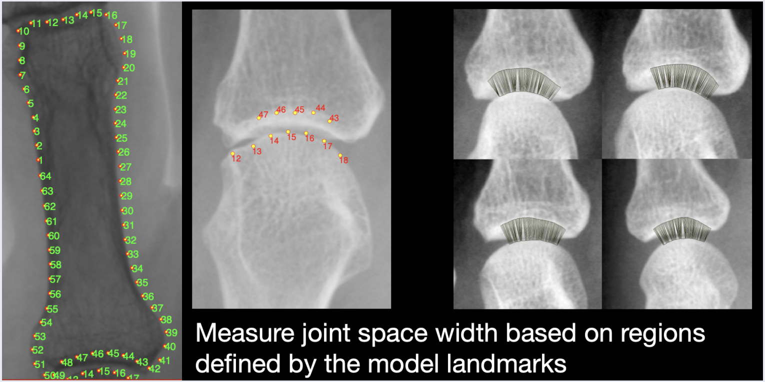


Figure: credit: [2]

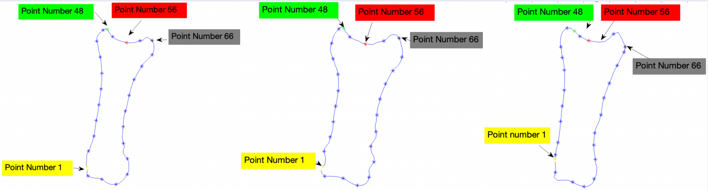
Rheumatoid Arthritis

To measure Joint Space, one uses landmarks along the contours of bones.
Difficulty : landmarks have to be placed at the same anatonical positions.



Rheumatoid Arthritis

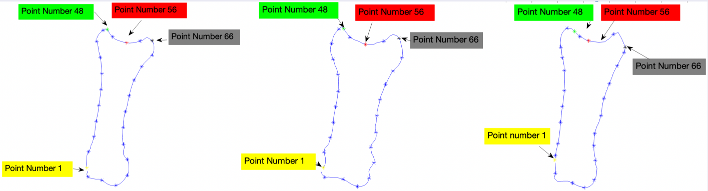
Difficulty : landmarks have to be placed at the same anatonical positions.
These positions have geometric characteristics depending on curvature.



Rheumatoid Arthritis

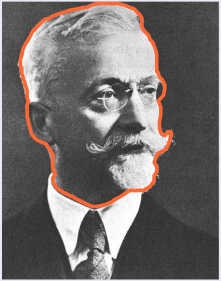
Task : automatically setup landmarks at correct anatonical positions.

To solve this task, we investigate curvature-dependant parameterizations of the contours of bones.



Curvature of a plane curves

Consider a plane simple closed curve. After the choice of a starting point and a direction, there is a unique way to travel the curve at unit speed : this is the arc-length parameterization. The rate of turning angle of the velocity vector is called the signed curvature of the curve. For instance, when moving along the external outline of the glasses, this curvature equals the inverse of the radius of the glasses.



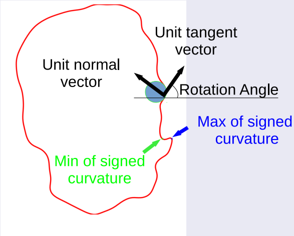


Figure: Elie Cartan and the moving frame associated to the contour of his head.

Curvature of a plane-Curve

The curvature function of Elie Cartan's head looks like this in different parameterizations:

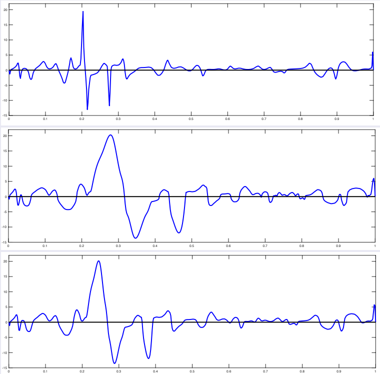


Figure: Signed curvature of Elie Cartan's head for the parameterization proportional to arc-length (first line), proportional to the curvature-length (second line), and proportional to the curvarc length (third line).

Arc-length Parameterization

It is easy to resample a curve using the arc-length parameterization : one computes the distances between points and resample uniformly using interpolation with splines.

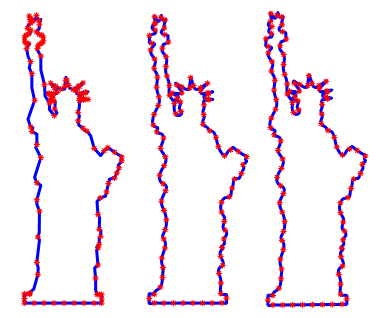


Figure: The statue of Liberty (left), a uniform resampling using Matlab function spline (middle), a reconstruction of the statue using its discrete curvature (right).

$\mathcal{C}_1(I)$ = length-one curves parameterized by $I = [0,1]$ or $\mathbb{S}^1$
$\mathscr{A}_1(I)$ = arc-length parameterized curves with length one

Theorem (A.B.T, S.Preston)

Given a curve $\gamma \in \mathcal{C}_1(I)$, let $p(\gamma) \in \mathscr{A}_1(I)$ denote its arc-length-reparameterization, so that $p(\gamma) = \gamma \circ \psi$ where

$$\psi'(s) = \frac{1}{|\gamma'(\psi(s))|}, \qquad \psi(0) = 0. \tag{1}$$

Then p is a smooth retraction of $\mathcal{C}_1(I)$ onto $\mathscr{A}_1(I)$.

Theorem (A.B.T, S.Preston)

$\mathscr{A}_1([0,1])$ is diffeomorphic to the quotient Fréchet manifold $\mathcal{C}_1([0,1])/\operatorname{Diff}^+([0,1])$.

Curvature-length Parameterization

Arc-length parameterization is used to compute the curvature function. One can integrate the absolute value of the curvature along the curve and renormalized to have a total integral equal to 1. The resulting function is used to define the curvature-length parameterization of the curve and resample the curve accordingly.

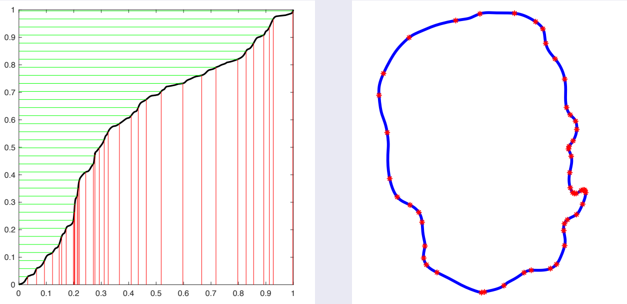


Figure: Integral of the (renormalized) absolute value of the curvature (left), and corresponding resampling of Elie's Cartan head (right).

From Curvature-length Parameterization to Arc-length Parameterization

The draw-back of curvature-length parameterization is that it does not put points at all on flat pieces of the curve. In order to fix this, instead of integrating the curvature, one can integrate λ + curvature, where λ is a parameter.

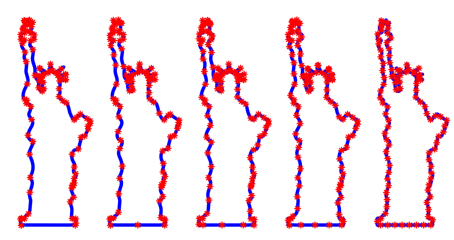


Figure:
Resampling of the statue of Liberty proportional to the intergral of λ + curvature, for (from left to right) $\lambda = 0$; $\lambda = 0.3$; $\lambda = 1$; $\lambda = 2$; $\lambda = 100$.

Curvarc-length Parameterization

The Curvarc-length parameterization is defined by $u(s) = \frac{\int_0^s (L+|\kappa(s)|)ds}{\int_0^1 (L+|\kappa(s)|)ds}$ where L is the length of the curve, and κ the curvature function.

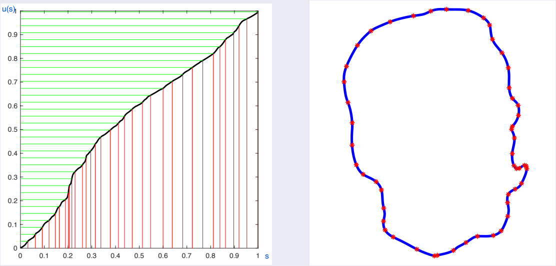


Figure: Integral of the (renormalized) curvarc length (left), and corresponding resampling of Elie's Cartan head (right).

WHAT TO REMEMBER SO FAR?

- ✓ the only invariant of plane curves is the curvature (in arc-length parameterization)
- ✓ Any strictly increasing function applied to the absolute value of the curvature defines a canonical parameterization of plane curves
- ✓ finding the right parameterization adapted to a given application = finding the right strictly increasing function

What is the right parameterization of bones?

We investigate a 2 parameter family of parameterizations defined by

$$u(s) = \frac{\int_0^s (c * L + |\kappa(s)|^\lambda) ds}{\int_0^1 (c * L + |\kappa(s)|^\lambda) ds} \quad (2)$$

where c and λ are positive parameters and where L is the length of the curve and κ its curvature function.

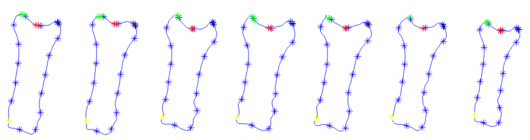
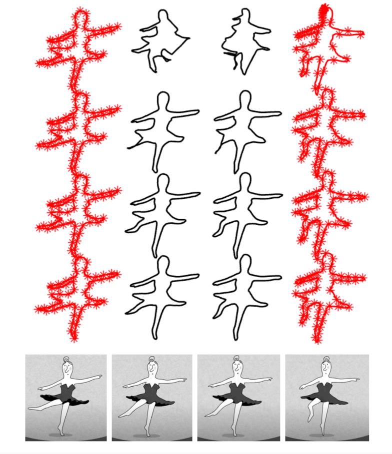


Figure: 7 bones parameterized by (2) with $c = 1$ and $\lambda = 7$. The colored points corresponds to points labelled 1, 48, 56, 66.

Shape spaces are non-linear manifolds



Parameterization of infinite-dimensional manifolds

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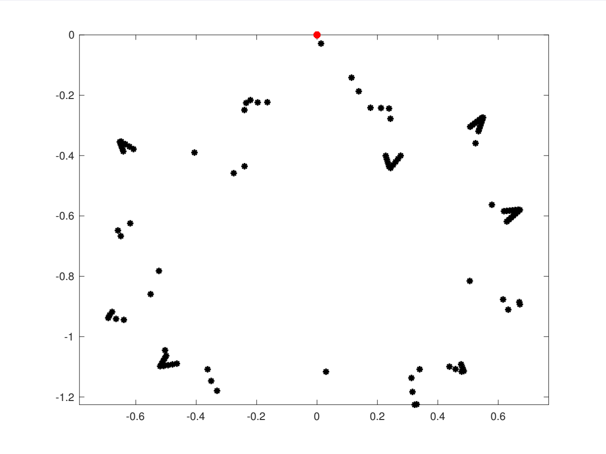

Figure: One leaf of Acer parameterized by the curvature

Parameterization of infinite-dimensional manifolds

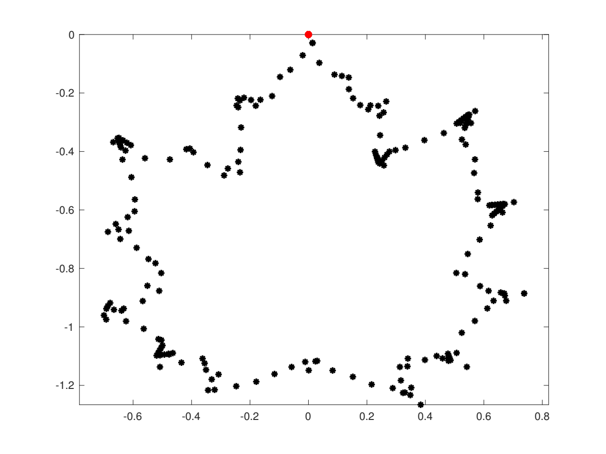

Figure: One leaf of Acer parameterized by the curvature and on top of it the same leaf parameterized by arc-length

Parameterization of infinite-dimensional manifolds

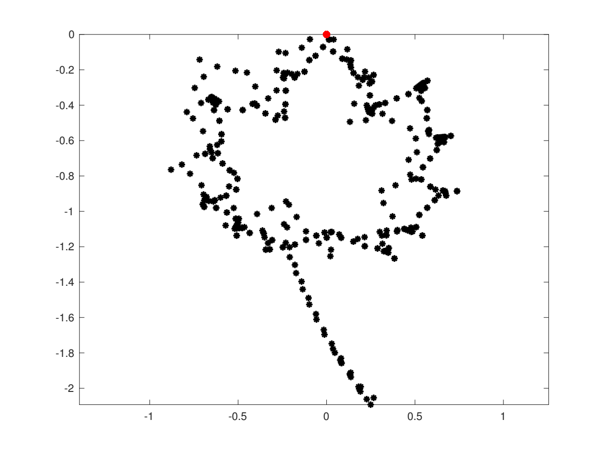

Figure: 2 leaves of Acer parameterized by arc-length

Parameterization of infinite-dimensional manifolds

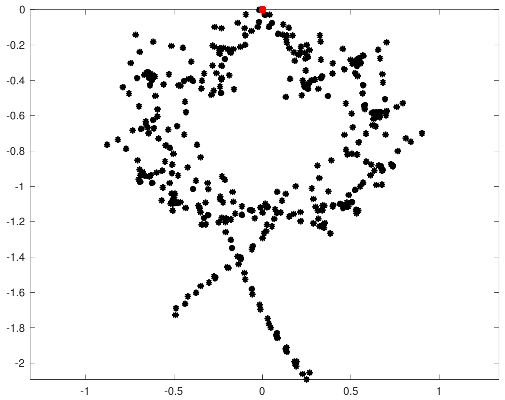

Figure: 3 leaves of Acer parameterized by arc-length

Parameterization of infinite-dimensional manifolds

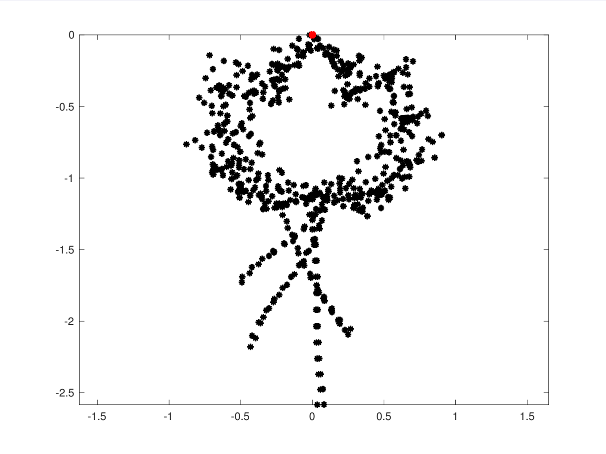

Figure: 5 leaves of Acer parameterized by arc-length

Parameterization of infinite-dimensional manifolds

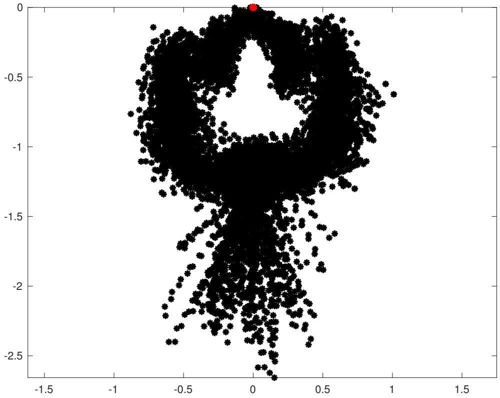

Figure: 75 leaves of Acer parameterized by arc-length

Parameterization of infinite-dimensional manifolds

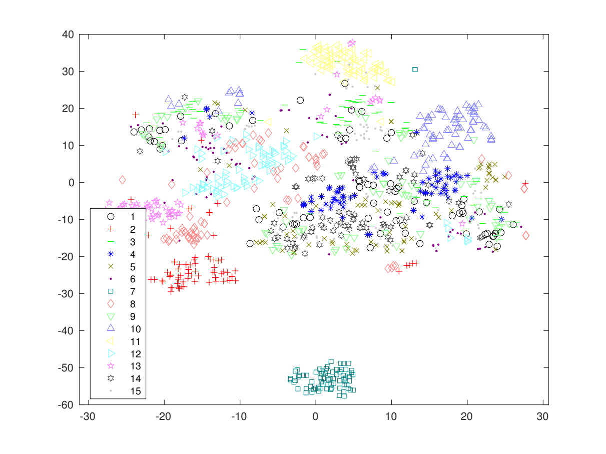

Figure: Distances between leaves of Acer parameterized by the curvature

Parameterization of infinite-dimensional manifolds

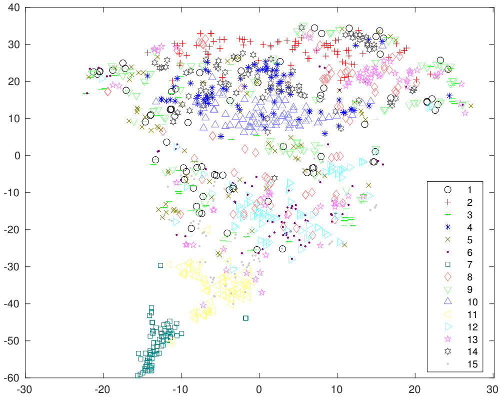

Figure: Distances between leaves of Acer parameterized by the curvature to third power

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