On the integrable deformations of continuous-time dynamical systems

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2023, Timisoara

A deformation of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} = (x, y, z),\ \mathbf{f} = (f_1, f_2, f_3) \tag{1}$$

is given by a system of the form

$$\begin{cases} \dot{x} = f_1(x, y, z) + u_1(x, y, z, \mathbf{d}) \\ \dot{y} = f_2(x, y, z) + u_2(x, y, z, \mathbf{d}) \\ \dot{z} = f_3(x, y, z) + u_3(x, y, z, \mathbf{d}) \end{cases}, \tag{2}$$

where the vector **d** contains the deformation parameters such that

if $\mathbf{d} \to \mathbf{0}$ *then* (2) $\to$ (1).

The functions added to a system by deformation can be viewed as control functions and as a consequence some natural queries arise:

- how does the stability change
- can be stabilized some states/orbits
- how is the periodic motion affected
- can be obtained new chaotic systems
- can be stabilized some chaotic trajectories

etc.

A function $F = F(x, y, z)$ is a constant of motion of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} = (x, y, z),\ \mathbf{f} = (f_1, f_2, f_3)$$

if

$$\dot{F} = \nabla F \cdot \mathbf{f} = 0.$$

The above system is integrable if there are two independent functions that are constants of motion of this system. In fact, such a system is a Hamilton-Poisson system.

A deformation of this system is an integrable deformation if it has two independent constants of motion too.

Integrable deformations for a three-dimensional Hamilton-Poisson system

Let H, C be independent constants of motion of the system

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}) \ , \ \mathbf{x} = (x, y, z), \mathbf{g} = (g_1, g_2, g_3). \tag{3}$$

It is known there is a function $\mu \in C^1$ (see, for example, [ ][1]) such that

$$\mathbf{g} = \mu \nabla H \times \nabla C.$$

Indeed, because H and C are constants of motion it follows

$$\nabla H \cdot \mathbf{g} = 0, \nabla C \cdot \mathbf{g} = 0 \Leftrightarrow \nabla H \perp \mathbf{g}, \nabla C \perp \mathbf{g}.$$

Therefore $\nabla H \times \nabla C$ and $\mathbf{g}$ are parallel, whence the conclusion follows.

[1] M. Gürses, G. S. Guseinov, K. Zheltukhin, *Dynamical systems and Poisson structures*, J. Math. Phys. 50 (2009) 112703.

Now, we consider the system

$$\dot{\mathbf{x}} = \mu \nabla \tilde{H} \times \nabla \tilde{C} \ , \quad \tilde{H}(\mathbf{x}) = H(\mathbf{x}) + k_1 u(\mathbf{x}), \ \tilde{C}(\mathbf{x}) = C(\mathbf{x}) + k_2 v(\mathbf{x}), \quad (4)$$

where u, v are arbitrary differentiable functions and $k_1, k_2 \in \mathbb{R}$.
An easy computation shows that system (4) becomes

$$\dot{\mathbf{x}} = \mu \nabla H \times \nabla C + k_2 \mu \nabla H \times \nabla v + k_1 \mu \nabla u \times \nabla C + k_1 k_2 \mu \nabla u \times \nabla v. \quad (5)$$

Remark. Because $\tilde{H}$ and $\tilde{C}$ are constants of motion of system (5), it is clear that this system is an integrable deformation of the integrable system $\dot{\mathbf{x}} = \mu \nabla H \times \nabla C$.

Consider the motion of a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} = (x, y, z),\ \mathbf{f} = (f_1, f_2, f_3)$$

in the phase space as the flow of a fluid. Hence the logarithmic rate of the volume change is expressed as[2]

$$\frac{1}{V}\frac{dV}{dt} = \nabla \cdot \mathbf{f},$$

where

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

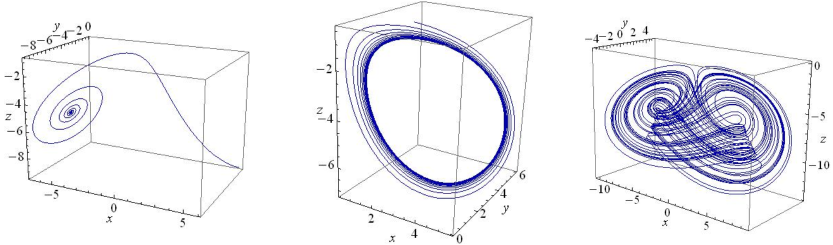
is the divergence of the vector field $\mathbf{f}$.

If the divergence is less than zero, then the system is called dissipative.

[2] E. N. Lorenz, Deterministic nonperiodic flow, J. Atmos. Sci. 20 (1963) 130–141.

A dissipative system has "a state space volume that decreases on average along the trajectory so that the orbit approaches an attractor of measure zero in the state space"[3].

For a three-dimensional dissipative dynamical system these attractors can be stable foci, limit cycles or strange chaotic attractors, and they can coexist.[4]



[3] J. C. Sprott, A dynamical system with a strange attractor and invariant tori, Phys. Lett. A 378 (2014) 1361–1363.

[4] J. C. Sprott, X. Wang, G. Chen, Coexistence of point, periodic and strange attractors, Int. J. Bifurcation and Chaos 23 (2013) 1350093-1–5.

The integrable deformations method for a three-dimensional system of differential equations

Let $\Omega \subseteq \mathbb{R}^3$ be an open set. We consider the following three-dimensional dynamical system on Ω

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \ , \ \mathbf{x} = (x, y, z), \mathbf{f} = (f_1, f_2, f_3). \tag{6}$$

We assume that system (6) has a Hamilton-Poisson part described by a vector field $\mathbf{g}$ (see, for example, Kolmogorov systems[5]). Therefore system (6) can be written in the form

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}) + \mathbf{h}(\mathbf{x}) \ , \tag{7}$$

where the system

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}) \tag{8}$$

has two functionally independent constants of motion, $H = H(\mathbf{x})$ and $C = C(\mathbf{x})$.

[5] V. I. Arnold, *Kolmogorov's hydrodynamic attractors*, Proc. Roy. Soc. Lond. A 434 (1991) 19–22.

As above, we construct integrable deformations of the Hamilton-Poisson part $\dot{\mathbf{x}} = \mu \nabla H \times \nabla C$ taking $\tilde{H} = H + \alpha$, $\tilde{C} = C + \beta$. Then, adding the non-conservative part $\mathbf{h}$, we obtain the integrable deformations of the given system[6] :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mu \nabla H \times \nabla \beta + \mu \nabla \alpha \times \nabla C + \mu \nabla \alpha \times \nabla \beta. \tag{9}$$

Remark. If the function μ is constant, then the divergence of system (9) is equal to the divergence of the initial system (6), $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

[6] C. Lăzureanu, *Integrable Deformations of Three-Dimensional Chaotic Systems*, Int. J. Bifurcat. Chaos 28 (5) (2018) 1850066.

• there is an integrable deformation of a given chaotic system that has also a chaotic behavior
• there is an integrable deformation of a nonchaotic version of a chaotic system that presents a chaotic attractor
• there is an integrable deformation which stabilizes an unstable state of a given system

Integrable deformations of the Lorenz system

The first chaotic attractor in a three-dimensional autonomous system is given by the Lorenz system[7]

$$\left\{\begin{array}{rcl} \dot{x} & = & a(y-x) \\ \dot{y} & = & cx - y - xz \\ \dot{z} & = & -bz + xy \end{array}\right., \ (a = 10, b = \frac{8}{3}, c = 28). \tag{10}$$

We consider the following Hamilton-Poisson part of system (10)

$$\left\{\begin{array}{rcl} \dot{x} & = & 0 \\ \dot{y} & = & cx - xz \\ \dot{z} & = & xy \end{array}\right., \tag{11}$$

with the constants of motion $H = \frac{1}{2}x^2$, $C = \frac{1}{2}y^2 + \frac{1}{2}z^2 - cz$.
System (11) writes $\dot{\mathbf{x}} = \nabla H \times \nabla C$.

[7] E. N. Lorenz, *Deterministic nonperiodic flow*, J. Atmos. Sci. 20 (2) (1963) 130–141.

Therefore we give the following integrable deformation of Lorenz system[8]

$$\begin{cases} \dot{x} &= a(y-x)+(z-c)\alpha_y - y\alpha_z + \alpha_y\beta_z - \alpha_z\beta_y \\ \dot{y} &= cx - y - xz - x\beta_z - (z-c)\alpha_x - \alpha_x\beta_z + \alpha_z\beta_x \\ \dot{z} &= -bz + xy + x\beta_y + y\alpha_x + \alpha_x\beta_y - \alpha_y\beta_x \end{cases} . \tag{12}$$

1. We deform a chaotic system and obtain another chaotic system.

Choosing the deformation functions $\alpha = \frac{g}{2}z^2$, $\beta = 0$, we obtain the system

$$\begin{cases} \dot{x} &= a(y-x) - gyz \\ \dot{y} &= cx - y - xz \\ \dot{z} &= -bz + xy \end{cases} , \tag{13}$$

where $g \in \mathbb{R}$ is the deformation parameter. We notice that if $g = 1$, then this system is so-called Qi system[9], which is a chaotic system.

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[8] C. Lăzureanu, *Integrable Deformations of Three-Dimensional Chaotic Systems*, Int. J. Bifurcat. Chaos Int. J. Bifurcat. Chaos 28 (5) (2018) 1850066.

[9] G. Qi, G. Chen, S. Du, Z. Chen, Z. Yuan, *Analysis of a new chaotic system*, Physica A 352 (2005) 295–308.

2. We deform a non-chaotic version of a chaotic system and obtain a chaotic system.

Setting $\alpha = 0$, $\beta = gxy$, system (12) becomes

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - y - xz \\ \dot{z} = -bz + xy + gx^2 \end{cases} . \tag{14}$$

The Lorenz system is not a chaotic one for some values of the parameters a, b, c. However, for the same values of a, b, c, we can choose deformation functions such that the obtained system has a chaotic behavior. For example, setting $a = 10, b = 5, c = 28$, the Lorenz attractor (10) is an asymptotically stable equilibrium point, and, if $g = 3$, the attractor of the integrable deformation of Lorenz system given by (14) is chaotic.

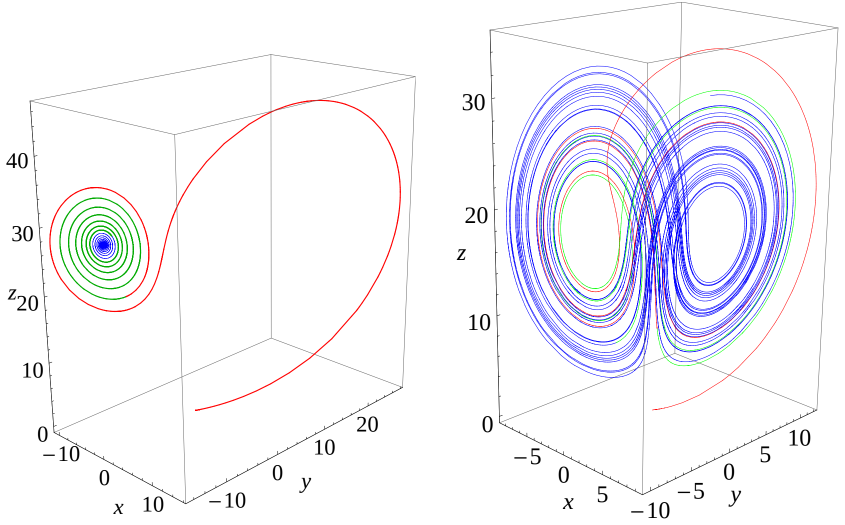

Figure: The Lorenz attractor (left) and one of its deformations (right, $g = 3$), for $a = 10, b = 5, c = 28$ ($t \in [0, 2]$ blue line, $t \in [2, 6]$ green line; initial conditions $x(0) = y(0) = z(0) = 0.1$)

Integrable deformations of Rössler system

In 1976, Rössler[10] proposed the following "prototype equation to the Lorenz model of turbulence which contains just one (second-order) nonlinearity"

$$\left\{\begin{array}{lcl} \dot{x} & = & -y - z \\ \dot{y} & = & x + ay \\ \dot{z} & = & -cz + xz + b \end{array}\right., \text{ where } a = 0.2, b = 0.2, c = 5.7. \tag{15}$$

We consider the following Hamilton-Poisson part of system (15)

$$\left\{\begin{array}{lcl} \dot{x} & = & -y \\ \dot{y} & = & x \\ \dot{z} & = & 0 \end{array}\right., \tag{16}$$

with the constants of motion $H = z,\ C = \frac{1}{2}x^2 + \frac{1}{2}y^2$. We get that $\mu = 1$.

[10] O. E. Rössler, *An equation for continuous chaos*, Phys. Lett. A 57 (1976) 397–398.

We give the following integrable deformation of Rössler system[11]

$$\begin{cases} \dot{x} &=& -y - z - \beta_y - y\alpha_z + \alpha_y\beta_z - \alpha_z\beta_y \\ \dot{y} &=& x + ay + \beta_x + x\alpha_z - \alpha_x\beta_z + \alpha_z\beta_x \\ \dot{z} &=& -cz + xz + b + y\alpha_x - x\alpha_y + \alpha_x\beta_y - \alpha_y\beta_x \end{cases} . \quad (17)$$

Considering the deformation functions $\alpha = \frac{g}{2}z^2$, $\beta = 0$, system (17) becomes

$$\begin{cases} \dot{x} &=& -y - z - gyz \\ \dot{y} &=& x + ay + gxz \\ \dot{z} &=& -cz + xz + b \end{cases} , \quad (18)$$

where the parameters a, b, c take the above-mentioned values, and $g \in \mathbb{R}$ is the deformation parameter.

[11] C. Lăzureanu, *Integrable Deformations of Three-Dimensional Chaotic Systems*, Int. J. Bifurcat. Chaos Int. J. Bifurcat. Chaos 28 (5) (2018) 1850066.

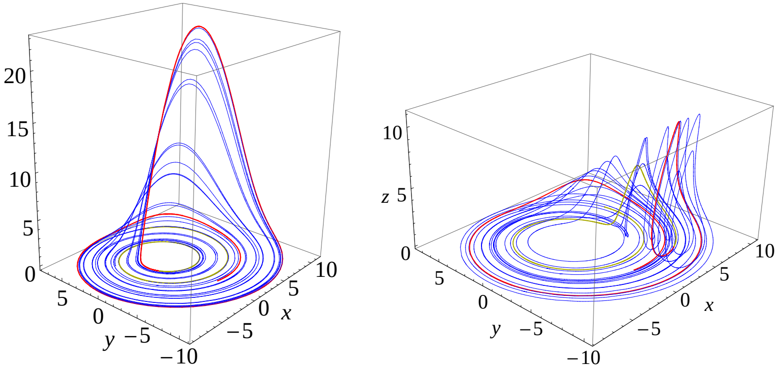

Figure: The Rössler attractor (left) and one of its deformations (right, $g = -0.25$), $a = 0.2$, $b = 0.2$, $c = 5.7$ ($t \in [0, 10]$ blue line, $t \in [10, 20]$ green line; initial conditions $x(0) = 0, y(0) = -6.78, z(0) = 0.02$)

A generalization: Integrable Deformations of the Maximally Superintegrable Systems

In classical mechanics, a superintegrable system on a $2n$-dimensional phase space is a completely integrable Hamiltonian system which possesses more functionally independent first integrals than degrees of freedom. Moreover, such a system is called maximally superintegrable[12] if the number of the independent first integrals is $2n-1$.
Similarly, for an arbitrary natural number n, a system of first-order differential equations on $\mathbb{R}^n$ which has $n-1$ functionally independent constants of motion is called a maximally superintegrable system.

If we can identify a maximally superintegrable part of the considered system, then we proceed as in the three-dimensional case.

[12] A.V. Tsiganov, *On maximally superintegrable systems*, Regul. Chaotic Dyn. 13 (2008) 178–190.

The functions $C_1, C_2, \ldots, C_{n-1}$ define the maximally superintegrable Hamilton-Poisson system

$$\dot{x}_k = \{H, x_k\}^{\nu}_{C_1,\ldots,C_{n-2}} , \quad k \in \{1, 2, \ldots, n\}, \tag{19}$$

with the Hamiltonian $H := C_{n-1}$ and the Poisson bracket[13] given by

$$\{f, g\}^{\nu}_{C_1,\ldots,C_{n-2}} := \nu \cdot \frac{\partial(C_1, C_2, \ldots, C_{n-2}, f, g)}{\partial(x_1, x_2, \ldots, x_{n-1}, x_n)} , \quad f, g \in C^{\infty}(\mathbb{R}^n, \mathbb{R}). \tag{20}$$

Conversely, for a given maximally superintegrable system, there is an unique function ν such that the system takes the form (19).

[13] R.M. Tudoran, *A normal form of completely integrable systems*, J. Geom. Phys. 62 (2012) 1167–1174.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in C^1(\Omega, \mathbb{R})$ be arbitrary functions such that the functions

$$I_k = C_k + g_k \alpha_k \ , \ k \in \{1, 2, \ldots, n-1\}$$

are functionally independent on an open set $\Omega \subseteq \mathbb{R}^n$, where $g_1, g_2, \ldots, g_{n-1} \in \mathbb{R}$. Then an integrable deformation of the above system is given by[14]

$$\begin{aligned} \dot{x}_k &= \{C_{n-1}, x_k\}^{\nu}_{C_1,\ldots,C_{n-2}} + g_1\{C_{n-1}, x_k\}^{\nu}_{\alpha_1, C_2\ldots,C_{n-2}} \\ &+ \sum_{i=2}^{n-3} g_i\{C_{n-1}, x_k\}^{\nu}_{I_1,\ldots,I_{i-1},\alpha_i,C_{i+1},\ldots,C_{n-2}} \\ &+ g_{n-2}\{C_{n-1}, x_k\}^{\nu}_{I_1,\ldots,I_{n-3},\alpha_{n-2}} + g_{n-1}\{\alpha_{n-1}, x_k\}^{\nu}_{I_1,\ldots,I_{n-2}} \ , \end{aligned}$$

$k \in \{1, 2, \ldots, n\}$.

[14] C. Lăzureanu, *On the Integrable Deformations of the Maximally Superintegrable Systems*, Symmetry 13 (2021) 1000.

Consider a Lotka-Volterra type system[15] given by

$$\dot{x}_1 = x_1 x_2 \ , \ \dot{x}_2 = x_2(x_3 - x_1) \ , \ \dot{x}_3 = x_3(x_4 - x_2) \ , \ \dot{x}_4 = -x_3 x_4,$$

with a constant of motion given by $H = x_1 + x_2 + x_3 + x_4$.
Apparently, this system is not maximally superintegrable. However, we can consider a maximally superintegrable part of it, namely

$$\dot{x}_1 = 0 \ , \ \dot{x}_2 = x_2 x_3 \ , \ \dot{x}_3 = -x_2 x_3 \ , \ \dot{x}_4 = -x_3 x_4.$$

Indeed, the functions $C_1 = x_1$, $C_2 = x_2 + x_3$, and $C_3 = x_2 x_4$ are constants of motion. Moreover, the above system takes the form

$$\dot{x}_k = \{C_3, x_k\}^{\nu}_{C_1, C_2} \ , \ k \in \{1, 2, 3, 4\},$$

where $\nu = x_3$.
Now we can construct integrable deformations.

[15] O.I. Bogoyavlenskij, *Integrable Lotka–Volterra Systems*, Regul. Chaotic Dyn. 13 (2008) 543–556.

Note that H is not a constant of motion for the deformed system, generally.
A particular deformations is obtained using

$$I_1 = C_1 + g\alpha(x_4), I_2 = C_2 + g\beta(x_4), I_3 = C_3,$$

namely

$$\dot{x}_1 = x_1x_2 + gx_3x_4\alpha'(x_4)\ ,\ \dot{x}_2 = x_2(x_3 - x_1),$$
$$\dot{x}_3 = x_3(x_4 - x_2) + gx_3x_4\beta'(x_4)\ ,\ \dot{x}_4 = -x_3x_4.$$

Therefore, if $\alpha(x_4) + \beta(x_4) = \mathit{constant}$, then the constant of motion $H = x_1 + x_2 + x_3 + x_4$ of the initial system is also a constant of motion of the deformed system.

Thank you! :)