Special classes of solutions to the Gross-Clark system

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Introduction

We consider the system

$$(\mathrm{GC}) \qquad \begin{cases} i\dfrac{\partial\Psi}{\partial t} + \Delta\Psi = \dfrac{1}{\varepsilon^2}\Psi\Big(|\Psi|^2 - 1 + \dfrac{|\Phi|^2}{\varepsilon^2}\Big) \\[2ex] i\delta\dfrac{\partial\Phi}{\partial t} + \Delta\Phi = \dfrac{1}{\varepsilon^2}\Phi(q^2|\Psi|^2 - \varepsilon^2 k_M^2). \end{cases}$$

- Introduced by Gross (1958) and Clark (1966) to describe the motion of an uncharged impurity in a Bose condensate.
- Ψ is the wave function of the condensate. When $\Phi = 0$, Ψ satisfies the Gross-Pitaevskii equation

$$(\mathrm{GP}) \qquad i\frac{\partial\Psi}{\partial t} + \Delta\Psi = \frac{1}{\varepsilon^2}\Psi\Big(|\Psi|^2 - 1\Big).$$

- Physical conditions:

$$|\Psi(t,x)| \longrightarrow 1 \quad \text{as} \quad |x| \longrightarrow \infty \qquad \text{and} \quad \int_{\mathbb{R}^N} |\Phi|^2(t,x)\,\mathrm{d}x < \infty.$$

Introduction

- Φ = wave function for the impurity
- Dimensionless constants introduced by the physicists:

δ = mass of the impurity / boson mass (small)

q^2 = boson-impurity scattering length / (2·boson diameter)

k_M = dimensionless measure for the single-particle impurity energy

$$\varepsilon^5 = \frac{\text{healing length}}{\text{boson-impurity scattering length}} \cdot \frac{\text{impurity mass}}{\text{boson mass}} \qquad \varepsilon \cong 0.2$$

- Previous work by Grant - Roberts (1974), N. Berloff - Roberts (2002-2006)...

The Gross-Pitaevskii equation

The **Gross-Pitaevskii** equation

(GP) $$i\frac{\partial \Psi}{\partial t} + \Delta \Psi = \frac{1}{\varepsilon^2}\Psi\Big(|\Psi|^2 - 1\Big)$$

and its stationary version, the **Ginburg-Landau** equation,

(GL) $$\Delta \Psi = \frac{1}{\varepsilon^2}\Psi\Big(|\Psi|^2 - 1\Big)$$

have been used as models for Bose-Einstein condensation, propagation of laser beams, liquid crystals, and received considerable attention during the last 30 years. The Ginzburg-Landau energy of Ψ is

(1) $$E_{GL}(\Psi) = \int_{\mathbb{R}^N} |\nabla \Psi|^2 + \frac{1}{2\varepsilon^2}(|\Psi|^2 - 1)^2 \, \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla \Psi|^2 + V(|\Psi|^2) \, \mathrm{d}x.$$

The natural function space for the study of (GL) and of (GP) is

$$\begin{aligned} \mathcal{E} &= \{\psi \in H^1_{loc}(\mathbb{R}^N) \mid E_{GL}(\psi) < \infty\} \\ &= \{\psi : \mathbb{R}^N \longrightarrow \mathbb{C} \mid \psi \text{ is measurable, } |\psi|^2 - 1 \in L^2(\mathbb{R}^N), \nabla \psi \in L^2(\mathbb{R}^N)\}. \end{aligned}$$

Conserved quantities and function space for (GC)

The energy of the (GC) system is

$$E(\Psi,\Phi) = \int_{\mathbb{R}^N} |\nabla\Psi|^2 + \frac{1}{2\varepsilon^2}(|\Psi|^2-1)^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\Phi|^2 + \frac{1}{\varepsilon^4}|\Psi|^2|\Phi|^2 \,\mathrm{d}x.$$

The mass of Ψ is

$$\mathbf{M}(\Phi) = \int_{\mathbb{R}^N} |\Phi|^2 \,\mathrm{d}x.$$

The energy and the mass are conserved by the flow associated to (GC).

It is natural to look for solutions $(\Psi,\Phi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$, where

$\mathcal{E}$ is the space of functions having finite Ginzburg-Landau energy,

$H^1(\mathbb{R}^N) = \{\varphi \in L^2(\mathbb{R}^N) \ | \ \nabla\psi \in L^2(\mathbb{R}^N)\}$.

Cauchy problem

Theorem (P. Gérard, 2006)

Let $N \in \{1, 2, 3\}$. For any $\Psi_0 \in \mathcal{E}$, the Gross-Pitaevskii equation has a unique global solution $\Psi : \mathbb{R} \longrightarrow \mathcal{E}$ such that $\Psi(0) = \Psi_0$.

Furthermore, the flow associated to (GP) is continuous and

$E_{GL}(\Psi(t)) = E_{GL}(\Psi_0)$ for all $t \in \mathbb{R}$.

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$E_{GL}(\Psi(t)) = E_{GL}(\Psi_0)$ for all $t \in \mathbb{R}$.

Theorem (J. Alhelou, 2021)

Assume that $N \in \{1, 2, 3\}$. For every $\Psi_0 \in \mathcal{E}$ and every $\Phi_0 \in H^1(\mathbb{R}^N)$ there exists a unique global solution (Ψ, Φ) of the (GC) system with initial values $(\Psi, \Phi)(0) = (\Psi_0, \Phi_0)$.

Moreover, the energy $E(\Psi(t), \Phi(t))$ and the mass $M(\Phi(t))$ are conserved by the flow associated to (GC).

Stationary solutions

Theorem

Let $N \geqslant 2$.

Any finite-energy solution of the Ginzburg-Landau equation in $\mathbb{R}^N$ is constant

Proof. Any solution $\psi \in \mathcal{E}$ is a critical point of E_{GL}. Let $\psi_\sigma(x) = \psi\left(\frac{x}{\sigma}\right)$.
Then $\frac{d}{d\sigma}_{|\sigma=1}(E_{GL}(\psi_\sigma)) = 0$ and this gives the Pohozaev identity

$$(N-2)\int_{\mathbb{R}^N} |\nabla\psi|^2\,dx + N\int_{\mathbb{R}^N} V(|\psi|^2)\,dx = 0$$

$\implies$ ψ is constant.

Stationary solutions for the (GC) system

We are looking for solutions of the (GC) system of the form

$$(\Psi, \Phi)(t, x) = \big(\psi(x), \mathrm{e}^{-i\omega t/\delta}\varphi(x)\big).$$

Then (ψ, φ) satisfy

$$(S_\omega) \qquad \begin{cases} -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi = 0 \\ -\Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k_M^2)\varphi = \omega \cdot \varphi \end{cases}$$

and are critical points of the *action* functional $E(\psi, \varphi) - \omega M(\varphi)$.

We are interested by *ground states* and we will consider the problem

$$\text{minimize } E(\psi, \varphi) \quad \text{ for } \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N) \text{ s.t. } \int_{\mathbb{R}^N} |\varphi|^2 \, dx = m.$$

For $m \geqslant 0$, we define

$$g_{min}(m) = \inf \left\{ E(\psi, \varphi) \mid \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |\varphi|^2 \, dx = m \right\}.$$

Recall that $E(\Psi, \Phi) = \int_{\mathbb{R}^N} |\nabla \Psi|^2 + \frac{1}{2\varepsilon^2}(|\Psi|^2 - 1)^2 + \frac{1}{\varepsilon^2 q^2}|\nabla \Phi|^2 + \frac{1}{\varepsilon^4}|\Psi|^2|\Phi|^2 \, \mathrm{d}x.$

Proposition

Assume that $N \in \{1, 2, 3\}$. Then:

(i) g_{min} is non-decreasing and concave on $(0, \infty)$, and $0 \leqslant g_{min}(m) \leqslant \frac{m}{\varepsilon^4}$ for all $m > 0$.

(ii) There exists $C > 0$ such that $g_{min}(m) \leqslant C m^{\frac{N}{N+2}}$.

(iii) If $N = 1$ we have $g_{min}(m) < \dfrac{m}{\varepsilon^4}$ for any $m > 0$ and $\lim\limits_{m \to 0} \dfrac{g_{min}(m)}{m} = \dfrac{1}{\varepsilon^4}$.

(iv) If $N \geqslant 2$, there exists $m_0(N) > 0$ such that $g_{min}(m) = \dfrac{m}{\varepsilon^4}$ for any $m \in (0, m_0(N)]$ and $g_{min}(m) < \dfrac{m}{\varepsilon^4}$ for $m \geqslant m_0(N)$.

Remark. We have $m_0(2) \leqslant 0.658$ and $m_0(3) \leqslant 4.61$.

Theorem

Assume that $g_{min}(m) < \frac{m}{\varepsilon^4}$. Then:

i) There exist minimizers for $g_{min}(m)$.

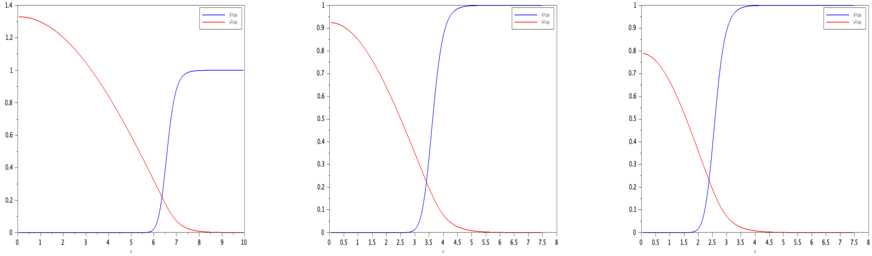
Moreover, all minimizing sequences are relatively compact (modulo translations).

ii) If $(\psi, \varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ is a minimiser, there exists $\gamma \in [g'_{min,r}(m), g'_{min,\ell}(m)]$ such that

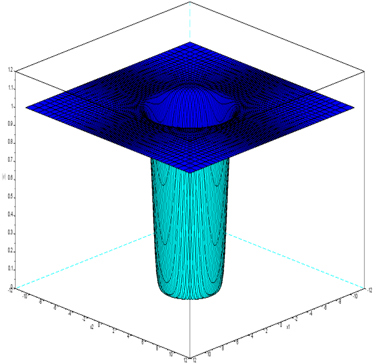
$$-\Delta\psi + F(|\psi|^2)\psi + \frac{1}{\varepsilon^4}|\varphi|^2\psi = 0,$$

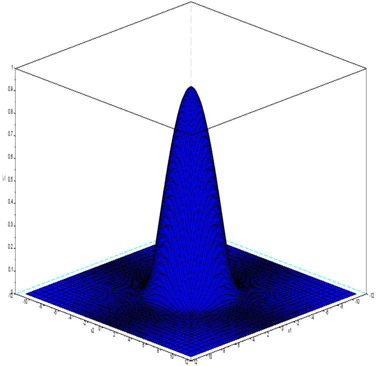
$$-\Delta\varphi + \frac{q^2}{\varepsilon^2}|\psi|^2\varphi - \varepsilon^2 q^2 \gamma\varphi = 0 \qquad \text{in } \mathbb{R}^N.$$

iii) The functions ψ and ϕ are smooth on $\mathbb{R}^N$ and after a phase shift, they are real-valued. After translation, they are radial. The radial profile of ψ is nondecreasing, and the radial profile of φ is nonincreasing.



Graphs of ψ_{GS} and φ_{GS} in radial coordinates with mass $\mathfrak{m} = 4\pi$ in dimension $N = 1$ (left) $N = 2$ (center) and $N = 3$ (right).



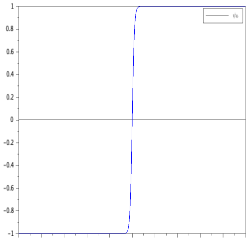


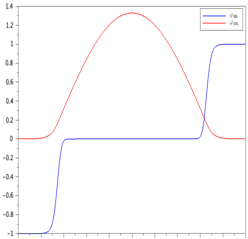
Stationary bubble-kink (1D)

In space dimension $N = 1$, the GP equation possesses some particular stationary solution known as the kink: $\psi_0(x) = \tanh\left(\frac{x}{\varepsilon\sqrt{2}}\right)$.

Theorem

Assume that $N = 1$ and that $m > 0$. Then, there exists $\omega \in \mathbb{R}$ and there is a least one solution (ψ, φ) to S_ω with ψ real-valued, odd and increasing from -1 to $+1$ and φ real-valued, even and decreasing in $\mathbb{R}_+$.





Stationary bubble-vortices (2D)

The Gross-Pitaevskii (GP) equation has some remarkable stationary solutions in the plane called *vortices* (Hervé and Hervé, 1994). These are stationary solutions which can be written in polar coordinates $x = r\cos\theta, \quad y = r\sin\theta$ in the form $\Psi(t,x) = \mathfrak{a}_d(r)e^{id\theta}$, where $d \in \mathbb{Z}^*$ is the winding number.

The profile $\mathfrak{a}_d : \mathbb{R}_+ \longrightarrow [0,1]$ solves the ODE

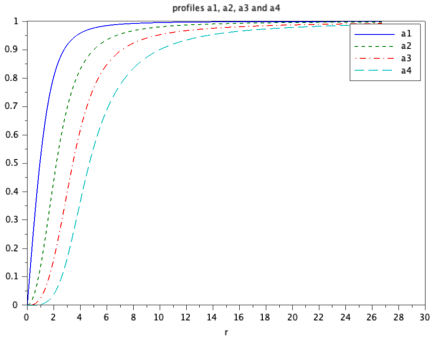
$$\mathfrak{a}_d''(r) + \frac{1}{r}\mathfrak{a}_d'(r) - \frac{d^2}{r^2}\mathfrak{a}_d(r) = \frac{1}{\varepsilon^2}\mathfrak{a}_d(r)\big(\mathfrak{a}_d^2(r) - 1\big)$$

and increases from 0 at $r = 0$ to 1 for $r \longrightarrow \infty$.

These solutions have infinite energy. Indeed, if $\psi(x) = \rho(x)e^{id\theta}$, we have $|\nabla\psi|^2 = |\nabla\rho|^2 + \frac{d^2}{r^2}\rho^2$. Therefore, if $\rho \longrightarrow 1$ as $|x| \longrightarrow \infty$, we get

$$\int_{B(0,R)} |\nabla\psi_d|^2\, dx \sim \int_{B(0,R)} |\nabla\rho|^2\, \mathrm{d}x + 2\pi d^2 \ln(R).$$

A graphical representation of $\mathfrak{a}_d$ for $1 \leqslant d \leqslant 4$ is as follows.



Theorem (H. Brezis - F. Merle - T. Rivière, 1994)

Let ψ be any solution of (GL) in $\mathbb{R}^2$ having topological degree d at infinity. Then

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\left(|\psi|^2-1\right)^2\,dx = 4\pi d^2.$$

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Theorem (P. Mironescu, 1996)

Let ψ be any solution of (GL) in $\mathbb{R}^2$ having topological degree 1 at infinity. There exists $x_0 \in \mathbb{R}^2$ such that $\psi(x-x_0) = a_1(|x|)e^{i\theta}$.

A similar result is unknown for solutions of degree $d > 1$ at infinity.

We look for stationary solutions of (GC) under the form

$$\big(\psi_d(x), \varphi_d(x)\big) = \big(\mathfrak{a}_d(r)e^{id\theta}, \mathfrak{f}_d(r)\big),$$

This yields to the system (in polar coordinates and with $\mathfrak{f}_d$ real-valued)

$$(SVB_{d,\omega}) \qquad \begin{cases} \mathfrak{a}_d'' + \dfrac{1}{r}\mathfrak{a}_d' - \dfrac{d^2}{r^2}\mathfrak{a}_d = \dfrac{1}{\varepsilon^2}\mathfrak{a}_d\Big(\mathfrak{a}_d^2 - 1 + \dfrac{\mathfrak{f}_d^2}{\varepsilon^2}\Big) \\ \\ \mathfrak{f}_d'' + \dfrac{1}{r}\mathfrak{f}_d' - \dfrac{1}{\varepsilon^2}\mathfrak{f}_d(q^2\mathfrak{a}_d^2 - \varepsilon^2 k_M^2 - \varepsilon^2\omega) = 0. \end{cases}$$

with the boundary conditions

$$\mathfrak{a}_d(0) = 0, \quad \mathfrak{a}_d(r) \longrightarrow 1 \quad \text{and} \quad \mathfrak{f}_d(r) \longrightarrow 0 \quad \text{as } r \longrightarrow \infty.$$

The mass constraint becomes

$$\int_{\mathbb{R}^2} \varphi_d^2\,dx = 2\pi\int_0^\infty \mathfrak{f}_d^2\,rdr = \mathfrak{m},$$

and ω depends on $\mathfrak{m}$ and possibly on $\mathfrak{f}_d$.

We need to renormalize the energy in order to deal with bubble-vortex solutions. We choose a cut-off function $\chi : [0, \infty) \longrightarrow [0, 1]$ such that $\chi = 0$ on $[0, 1]$, χ is non-decreasing, C^∞, and $\chi = 1$ on $[2, \infty)$. We consider

$$E_d(\rho, \varphi) = \int_{\mathbb{R}^2} |\nabla \rho|^2 + \frac{d^2}{|x|^2} \left(\rho^2(x) - \chi^2(|x|)\right) + \frac{1}{2\varepsilon^2}(|\rho|^2 - 1)^2$$

$$+\frac{1}{\varepsilon^2 q^2}|\nabla \varphi|^2 + \frac{1}{\varepsilon^4}|\rho|^2|\varphi|^2 \, \mathrm{d}x.$$

We study the problem

$$\text{minimize } E_d(\rho, \varphi) \quad \text{ for } \rho \in \mathcal{E},\ \phi \in H^1(\mathbb{R}^2) \text{ s. t. } M(\phi) = \mathfrak{m}.$$

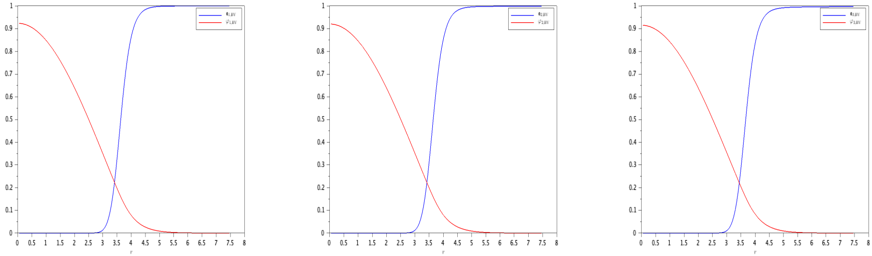
Theorem

Let $d \in \mathbb{N}^*$. Then:

i) For any $\mathfrak{m} > 0$ the above minimization problem admits at least a solution.

ii) If (ρ, φ) is a solution, then ρ and φ are smooth and radially symmetric, and φ is real-valued after a phase shift. The radial profile of ρ is non-decreasing, the radial profile of ϕ is non-increasing, and they solve the system $(SVB_{d,\omega})$ for some $\omega \in \mathbb{R}$.

Graphs of bubble-vortices with mass $\mathfrak{m} = 4\pi\ \mathfrak{a}_d$ (blue) and φ_d (red) in radial coordinate for $d = 1$ (left), $d = 2$ (center), $d = 3$ (right):



Traveling waves for (GP)

Traveling waves are solutions of (GC) of the form

$\Psi(x,t) = \psi(x_1 - ct, x_2, \ldots, x_N)$. The profile ψ satisfies

$$-ic\frac{\partial \psi}{\partial x_1} = -\Delta\psi + \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\phi|^2 - 1)\psi$$

Traveling waves of speed c for (GP) are critical points of the functional $E_{GL}(\psi) - cQ(\psi)$. These solutions have received a lot of attention (Grant-Roberts '74, Bethuel-Saut Ann IHP '99, Bethuel-Orlandi-Smets JFA '04, Bethuel-Gravejat-Saut CMP '09, M. '13, Chiron-M. ARMA '17, ...).

Theorem (P. Gravejat, 2003)

The (GP) equation does not admit non-constant finite energy traveling waves of speed $|c| > v_s = \sqrt{2}$.

Here $v_s = \sqrt{2}$ is the *sound velocity at infinity* for (GP).

The momentum

The momentum (with respect to x_1) is a functional Q such that $Q'(u) = 2iu_{x_1}$.

(a) If $u \in H^1(\mathbb{R}^N)$ or if $u \in a + H^1(\mathbb{R}^N)$ we have $Q(u) = \int_{\mathbb{R}^N} \langle iu_{x_1}, u \rangle \, dx$.

(b) If $\psi \in \mathcal{E}$ has a lifting $\psi = \rho e^{i\theta}$, we have (formally)

$$Q(\psi) = -\int_{\mathbb{R}^N} \rho^2 \theta_{x_1} \, dx = -\int_{\mathbb{R}^N} (\rho^2 - 1)\theta_{x_1} \, dx.$$

Using a functional analysis argument, we can define the momentum for any function $\psi \in \mathcal{E}$ in such a way that this definition agrees with (a) and (b).

The momentum is conserved by the Gross-Pitaevskii equation.

Let $c \in (-v_s, v_s)$. We are looking for critical points of the functional $E_{GL} - cQ$.

Three methods have been used :

- Minimize E_{GL} when Q is kept fixed, c will be a Lagrange multiplier $\Longrightarrow$ a family $\mathcal{T}_1$ of travelling waves (+ orbital stability).
- Minimize $E - Q$ when $\int_{\mathbb{R}^N} |\nabla \psi|^2 \, dx = const. \Longrightarrow$ a family $\mathcal{T}_2$
- Minimize $E - cQ$ under a Pohozaev constraint $\Longrightarrow$ a family $\mathcal{T}_3$

We have $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_3$.

Minimization of energy at fixed momentum.

Assume that $N = 2, 3$. For $p \geq 0$, let

$$E_{1,min}(p) = \inf\{E_{GL}(\psi) \mid \psi \in \mathcal{E}, \ Q(\psi) = p\}. \tag{2}$$

Theorem

We have:

(i) The function $E_{1,min}$ is concave, increasing on $[0,\infty)$, $E_{1,min}(p) \leq v_s q$ for any $q \geq 0$, the right derivative of E_{min} at 0 is v_s, $E_{min}(p) \longrightarrow \infty$ and $\frac{E_{1,min}(p)}{p} \longrightarrow 0$ as $p \longrightarrow \infty$.

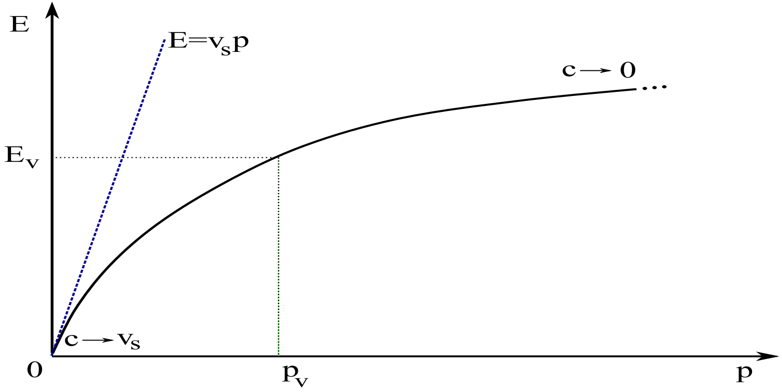
(ii) Let $p_0 = \inf\{p > 0 \mid E_{1,min}(p) < v_s p\}$. For any $p > p_0$, all sequences $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ satisfying $Q(\psi_n) \longrightarrow p$ and $E(\psi_n) \longrightarrow E_{min}(p)$ are precompact (modulo translations).

The set $\mathcal{S}_p = \{\psi \in \mathcal{E} \mid Q(\psi) = p,\ E(\psi) = E_{1,min}(p)\}$ is not empty and is orbitally stable by the flow associated to (GP).

(iii) Any $\psi_p \in \mathcal{S}_p$ is a traveling wave for (GP) of speed $c(\psi_p) \in [d^+E_{1,min}(p), d^-E_{1,min}(p)]$, where we denote by d^- and d^+ the left and right derivatives. We have $c(\psi_p) \longrightarrow 0$ as $p \longrightarrow \infty$.

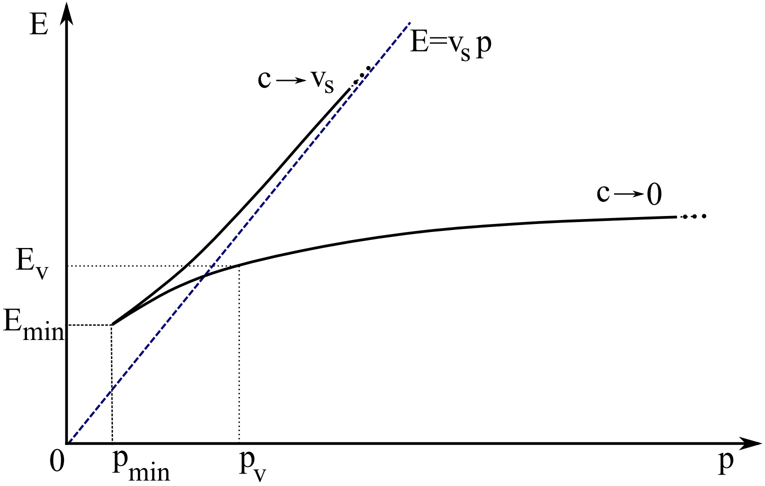
(iv) We have $p_0 = 0$ if $N = 2$ and $p_0 > 0$ if $N = 3$.

Energy-momentum diagram for (GP) in 2D



Energy-momentum diagram for traveling waves to (GP) in dimension 2.

Energy-momentum diagram for (GP) in 3D



Energy-momentum diagram for traveling waves to (GP) in dimension 3.

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Traveling waves are solutions of (GC) of the form

$\Psi(x,t) = \psi(x_1 - ct, x_2, \ldots, x_N)$, $\Phi(x,t) = \tilde{\varphi}(x_1 - ct, x_2, \ldots, x_N)$.

It is more interesting to search for $\tilde{\varphi}$ of the form $\tilde{\varphi}(x) = e^{i\delta c x_1}\varphi(x)$; this transform leads finally to $\Phi(x,t) = e^{i\delta c(x_1 - ct)}\varphi(x_1 - ct, x_2, \ldots, x_N)$. We find that ψ and ϕ must satisfy the system

$$(TW) \quad \left\{ \begin{array}{rcl} -2ic\dfrac{\partial \psi}{\partial x_1} & = & -\Delta\psi + \dfrac{1}{\varepsilon^2}(|\psi|^2 + \dfrac{1}{\varepsilon^2}|\phi|^2 - 1)\psi \\ \\ (\delta^2 c^2 + k^2)\varphi & = & -\Delta\phi + \dfrac{q^2}{\varepsilon^2}|\psi|^2\varphi. \end{array} \right.$$

The sound velocity at infinity associated to (GC) is $v_s = \frac{\sqrt{2}}{\varepsilon}$.

Theorem

Any traveling wave $(\psi, \varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ of speed $|c| > v_s$ is constant.

- $Q(\psi)$ and $Q(\varphi)$ are **not** conserved quantities for (GC). Let

$$P(\psi, \varphi) = Q(\psi) + \frac{\delta}{\varepsilon^2 q^2} Q(\varphi). \tag{3}$$

  It is easily seen that P is (at least formally) a conserved quantity for the system (GC). Therefore it is natural to seek for traveling waves for (GC) by minimizing E while P is kept fixed.

- Traveling waves of speed c for the system (CG) are critical points of the functional $E - cP$.

- Assume that (ψ, φ) is a critical point of $E - cP$, that is $d(E - cP)(\psi, \varphi) = 0$. There is an interplay between the mass and the momentum of φ: evaluating $d(E - cP)(\psi, \varphi).(0, ix_1\varphi)$ and integrating by parts we get

$$Q(\varphi) = \frac{c\delta}{2} \int_{\mathbb{R}^N} |\varphi|^2 \, dx.$$

We may proceed similarly for the (GC) system as for the (GP) equation and we consider the minimization problem

$$(\mathcal{P}_p) \qquad \text{minimize } E(\psi,\varphi) \text{ for } \psi \in \mathcal{E},\ \varphi \in H^1(\mathbb{R}^N) \text{ satisfying } P(\psi,\varphi) = p.$$

Let

$$E_{min}(p) = \inf \left\{ E(\psi,\varphi) \mid \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), P(\psi,\varphi) = p \right\}.$$

Proposition

Assume that $N \in \{2,3\}$. Then:

i) E_{min} is concave, positive and increasing on $(0,\infty)$, and $E_{min}(p) \longrightarrow \infty$, $\frac{E_{min}(p)}{p} \longrightarrow 0$ as $p \longrightarrow \infty$.

ii) There is $S_1 > 0$, explicitly depending on the physical parameters in (GC), such that $\lim_{p \to 0} \frac{E_{min}(p)}{p} = S_1$ and $E_{min}(p) \leqslant S_1 p$ for all $p > 0$.

If $N = 2$ we have $E_{min}(p) < S_1 p$ for all $p > 0$.

Let $p_0 = \inf\{p > 0 \mid E_{min}(p) < S_1 p\}$.

Theorem

Assume that $N = 2$ or $N = 3$, and $p > 0$ is such that $E_{min}(p) < S_1 p$.

Then there exist minimizers for the problem $(\mathcal{P}_p)$.

Moreover, any minimizing sequence $(\psi_n, \varphi_n)_{n \geqslant 1} \subset \mathcal{E} \times H^1(\mathbb{R}^N)$ contains a convergent subsequence (after translation).

Any minimizer ψ, ϕ of $(\mathcal{P}_p)$ solves the (TW) system for some $c \in [d^+ E_{min}(p), d^- E_{min}(p)]$.

The functions ψ and φ are smooth in $\mathbb{R}^N$ and axially symmetric about Ox_1 (after translation).

Minimization of the energy at fixed mass and momentum

We consider the problem

$$(\mathcal{E}_{p,m}) \qquad \text{minimize } E(\psi,\varphi) \text{ when } Q(\psi) = p \text{ and } \int_{\mathbb{R}^N} |\varphi|^2\,dx = m.$$

If (ψ, φ) is a minimizer, the parameters c and $\lambda = \delta^2 c^2 + k_M^2$ appearing in (TW) will be the corresponding Lagrange multipliers. For $p \in \mathbb{R}$ and $m \geqslant 0$, let

$$E_{min}(p,m) = \inf\left\{ E(\psi,\varphi) \;\middle|\; \psi \in \mathcal{E},\ \varphi \in H^1(\mathbb{R}^N),\ \begin{array}{l} Q(\psi) = p, \text{ and} \\ \int_{\mathbb{R}^N} |\varphi|^2\,dx = m \end{array} \right\}.$$

Recall that

$$E_{1,min}(q) = \inf\left\{ E_{GL}(\psi) \mid \psi \in \mathcal{E},\ Q(\psi) = q \right\}.$$

Proposition

The function E_{min} has the following properties:

(i) $E_{min}(p, m) = E_{min}(-p, m)$ for any $p \in \mathbb{R}$ and any $m \geqslant 0$.

(ii) $E_{min}(p, m)$ is finite and continuous on $\mathbb{R} \times [0, \infty)$, and for all $p \in \mathbb{R}$ and $m \geqslant 0$ we have $E_{min}(p, 0) = E_{1,min}(|p|)$, $E_{min}(0, m) = g_{min}(m)$, and

$$\max(E_{1,min}(|p|), g_{min}(m)) \leqslant E_{min}(p, m) \leqslant E_{1,min}(|p|) + g_{min}(m).$$

(iii) E_{min} is sub-additive:
$E_{min}(p_1 + p_2, m_1 + m_2) \leqslant E_{min}(p_1, m_1) + E_{min}(p_2, m_2)$ for all p_1, p_2, m_1, m_2.

(iv) For any fixed p_0 the mapping $m \longmapsto E_{min}(p_0, m)$ is concave and increasing on $[0, \infty)$.

(v) If $N \geqslant 3$, for any pair $(p_0, m_0) \neq (0, 0)$, $m_0 \geqslant 0$, the mapping $t \longmapsto E_{min}(tp_0, tm_0)$ is concave and increasing on $[0, \infty)$.

(vi) Assume that $p_1, p_2 \in \mathbb{R}$ and $m_1, m_2 \geqslant 0$ are such that

$$E_{min}(p_1, m_1) + E_{min}(p_2, m_2) = E_{min}(p_1 + p_2, m_1 + m_2).$$

Then we have either

$$E_{min}(p_1, 0) + E_{min}(p_2, m_1 + m_2) = E_{min}(p_1 + p_2, m_1 + m_2), \qquad \text{or}$$

$$E_{min}(p_1, m_1 + m_2) + E_{min}(p_2, 0) = E_{min}(p_1 + p_2, m_1 + m_2).$$

Theorem

Assume that $N = 2$ or $N = 3$ and the pair (p, m) satisfies the following strict sub-additivity condition:

$$(4) \qquad E_{1,min}(p') + E_{min}(p - p', m) > E_{min}(p, m) \text{ for any } p' \in \mathbb{R}^*.$$

Then the minimization problem $(\mathcal{E}_{p,m})$ admits solutions, and any minimizing sequence has a convergent subsequence (after translations).

Let

$$\mathcal{S} = \{(p, m) \in (0, \infty)^2 \mid (p, m) \text{ satisfies } (4) \}.$$

We are able to show that $\mathcal{S} \neq \emptyset$ (and in fact $\mathcal{S}$ is quite large).

We have checked numerically that some physically relevant pairs (p, m) belong to $\mathcal{S}$.

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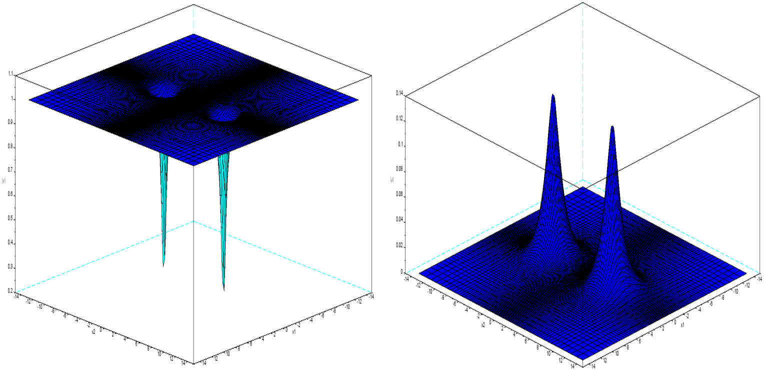
Question: Is it true that $(p, m) \in \mathcal{S}$ for all $p > p_0$ and $m > m_0$?

Small mass, high momentum traveling wave for (GC) in 2D

Graphs of ψ (left) and of φ (right):



Vă mulțumesc pentru atenție !